CAMBRIDGE UNIVERSITY PRESS
London: FETTER LANE, E C
C F CLAY, MANAGER

Edinburgh 100, PRINCES STREET
Berlin A ASHER AND CO
Leipzig F A BROCKHAUS
New York G P PUTNAM'S SONS
Bombay and Calcutta MACMILLAN AND CO, LTD

AN ELEMENTARY TREATISE

ON

CROSS-RATIO GEOMETRY

WITH HISTORICAL NOTES

BY THE

REV. JOHN J. MILNE, M.A.

ST JOHN'S COLLEGE, CAMBRIDGE.

AUTHOR OF *WEEKLY PROBLEM PAPERS*, &c.

Cambridge:

at the University Press

1911

Cambridge:
PRINTED BY JOHN CLAY, M.A.
AT THE UNIVERSITY PRESS.

PREFACE

THE development of the theory of cross-ratio is due, quite independently of each other[1], to Möbius, *Der barycentrische Calcul*, 1827, and to Chasles, *Aperçu Historique*, 1829—1837, followed by the *Géométrie Supérieure*, 1852, where the subject is treated very fully as regards the point and straight line, its application to the conic being given in the *Traité des Sections Coniques*, 1865. Some employment of its principles is met with in the various treatises on what is sometimes called Modern Geometry which have subsequently appeared, but as far as I am aware there is no English text-book exclusively devoted to it.

The power of the method of cross-ratio, as an instrument of analysis, it is not easy to over-rate. In the facility with which it deals alike with the range and pencil, with the points and line at infinity, with questions relating to concurrency and collinearity, loci and envelopes, it can compare not unfavourably with the methods of analytical geometry, and in those questions to which it is specially applicable, the steps necessary to establish any result are few in number, and are mostly of the same character, dealing as a rule with the homography of certain ranges or pencils, with the additional advantage that the geometrical meaning of each step is in general obvious.

[1] See the note on p. xxxii of the Preface to Chasles' *Géométrie Supérieure*, where in speaking of the Calcul barycentrische he says "ce que je n'ai su que fort longtemps après la publication de l'Aperçu historique."

Again, in dealing with pairs of imaginary points, analytical geometry is generally content with the recognition of their occurrence owing to certain relations between the coefficients of an equation; but the theory of cross-ratio goes further, and not only gives us the geometrical conditions under which they occur, but it gives us the actual position of their mid-point, and the value of the rectangle formed by the segments joining them to a real point.

This treatise naturally divides itself into two parts. In Chapters I—X, which deal exclusively with the point and straight line, the only knowledge of geometry which the reader is assumed to possess is that of the fundamental properties of similar triangles and ratio, and I have thought it advisable to make this part of the subject quite self-contained. It is usual to discuss co-axial ranges by projecting them on to a conic or circle, and making use of the Pascal line, &c., but by means of Prof. A. Lodge's method, given in Chap. VII, a student is enabled to construct two co-axial ranges, and to find their common points, &c., without interrupting the logical course of his reading.

In dealing with involution it seemed most simple and natural to treat it as the case of two co-axial homographic rows in which I and J', the correspondents of points at infinity, coincide.

In the second part, beginning with Chap. XI, I have adopted B. W. Horne's method of applying the theory of cross-ratio to the conic, which obviates the necessity of first proving properties for the circle, and then by projection obtaining the corresponding properties for the conic. This requires the knowledge on the part of the student of four elementary propositions in geometrical conics, viz. those given in Arts. 127, 128, 135, 136, and I have had no hesitation in assuming them for two reasons. In the first place, this part of the work is intended to be a treatise, not on geometrical conics, but on the application of the theory of cross-ratio to the subject; and secondly, although the subject of geometrical conics can be developed by means of the theory of cross-ratio, as Chasles has shewn in his fascinating *Traité des*

Sections Coniques, I am strongly of opinion that a student ought to have obtained from one of the ordinary text-books a working knowledge of its elements before he is introduced to the theorems of Pascal, Brianchon, Desargues, &c., which take him at once to more advanced work. Another reason is that by the aid of the theory of cross-ratio it is just as easy to prove properties of conics, considered separately or as a system, as it is to prove the corresponding properties for the circle, in fact in some cases it is easier and more complete, as we might expect from the consideration that the circle is only a particular case of the conic.

As there is scarcely any part of conics to which the theory of cross-ratio is not applicable, and as I wished to curtail the size of the book as much as possible, it was necessary to follow some definite path, and I have selected the course which leads us to consider conics through four points, and conics touching four lines, their common chords and tangents, the relations between the four constants of homology obtained by taking any pair of common chords with the pair of corresponding tangent vertices as axes and centres of homology, and conics having double contact. I took this route because it contains parts of the subject which have not previously been fully treated, and at the same time it gives the student a good illustration of the power of the theory. A good deal of the work in these chapters is original, and where it is not so, references have been given, where possible, to the original authorities.

I have thought it advisable to give a figure with almost every proposition so that the student may be enabled readily to follow all that he reads, and to remove any feeling of indefiniteness in his mind I have given full solutions in the case of problems which depend on finding the common points of two co-axial ranges. With the same object I have given complete figures in the different cases of the real and ideal common chords of two conics, and their common self-conjugate triangle.

The reader will notice that throughout the work I have made

no use either of projection, except in Arts. 138, 139, and in Chap. XIX (which deals with generalised projection), or of the principle of duality. My reason for the omission of the first is that it was not necessary for my purpose, and with regard to the latter the direct demonstration of a correlative theorem gave me an additional opportunity of illustrating the use of the theory of cross-ratio.

I have given at intervals throughout the work historical notes illustrative of the subject as far as it was in my power to do so within the limits of a private library, and by means of books kindly lent from the library of St John's College, Cambridge, and in doing this, one of my objects has been to shew that both parts of the subject are based upon ancient geometry, the theorem that a pencil cuts all transversals in equicross ratios being given by Pappus, and the converse of the anharmonic property of conics being due to Apollonius. With the same purpose in view I have given in an Appendix Pappus' account of the lost books of Euclid's Porisms, so that the student may have the opportunity of forming an opinion as to the probability of their connection with the theory of cross-ratio.

As the term "Modern Geometry" is frequently used without it being stated whether the adjective refers to the matter or the methods employed, or both, the following brief statement respecting the text-books on geometry in common use by the ancients will give the reader a general idea of the amount of knowledge of the subject which they possessed.

Pappus, in the preface to the seventh book of his Mathematical Collections, tells us that when a student had read the Elements of Euclid, and wished to proceed to more advanced work, the following was the order of the books which he would take up.

I. *Euclid's Data*, one book containing 100 theorems.

This is still extant, and to be met with in some of the older editions, *e.g.* that by Barrow 1732, and by R. Simson 1841.

The following works by Apollonius :

II. *Proportional Section*, two books containing 181 theorems.

This was discovered in an Arabic MS in the Bodleian Library, and a Latin translation was published by Halley in 1706. See Art. 88.

III. *Spatial Section*, two books containing 124 theorems.

This has been "restored" by Halley, and published with the books on Proportional Section. Another restoration was made by Snell 1607.

IV. *Determinate Section*, two books containing 83 theorems.

This has been restored by various geometers. Snell 1601, Lawson 1772, Wales 1772, Simson 1776.

V. *Tangencies*, two books, 81 theorems.

Restored by Vieta 1600, Lawson 1771.

VI. *Euclid's Porisms*, three books. See Appendix I.

The following by Apollonius :

VII. *Inclinations*, two books, 125 theorems.

This was a treatise respecting lines which pass through a given point whilst satisfying certain conditions (*e.g.* through a given point to draw a straight line such that the part of it intercepted between two given straight lines may be of given length). Restored by Ghetaldus 1607, Horsley, 1770, Burrow 1779.

VIII. *Plane loci*, two books, 147 theorems.

Restored by Schooten 1656, Fermat 1679, Simson 1749.

IX. *Conics*, eight books, 487 theorems.

Books I—IV are extant in Greek, V—VII were discovered in Arabic and translated into Latin by Ecchellensis and Borellus in 1661. In 1710 Halley published the first four books in Greek

and Latin, and the next three in Latin, together with a conjectural restitution of the 8th book, which is still missing[1].

X. Aristaeus, solid loci, five books.

XI. Euclid, loci ad superficiem.

XII. Eratosthenes, on Means, two books.

Of these, all except I, II and IX are lost, although it is quite possible they may still be in existence, probably in Arabic.

I have purposely refrained from giving a large number of Examples, and those given (260) have been carefully chosen to illustrate the text. The student who requires more will find admirable collections in Russell's *Elementary Treatise on Pure Geometry* (1905), and in Durell's *Plane Geometry for Advanced Students*, Part II (1910).

I take this opportunity of acknowledging my personal obligations to Prof. A. Lodge, of Charterhouse (late Professor of Pure Mathematics at Cooper's Hill), for the stimulating interest he has taken in the book throughout. He read through the whole of the work in manuscript, and again when it was passing through the press, and was of the greatest help in discussing the difficulties which arose from time to time; in fact he could not have taken a greater interest in it if the work had been his own, and it is chiefly owing to his friendly persistence that the treatise, which was originally written to gratify my own interest in the subject, has seen the light.

I am also greatly indebted to Prof. Heawood, of Durham, who kindly read through the proof-sheets, and from whom I received many valuable criticisms and suggestions.

[1] For a fuller account, see the *Math. Gazette* for October 1895.

JOHN J. MILNE.

LEE-ON-THE-SOLENT.

September, 1911.

CONTENTS

CHAPTER I

Cross-ratio of a range of four points, and of a pencil of four lines

CHAPTER II

Equicross ranges and equicross pencils. Perspective

CHAPTER III

Harmonic ratio

CHAPTER IV

Homographic ranges and homographic pencils

CHAPTER V

Cross-axis. Cross-centre

CHAPTER VI

Metrical properties of homographic ranges. The constant of correspondence. Homographic equations. One-to-one correspondence

CHAPTER VII

Two homographic co-axial ranges. Their common points, and the methods of finding them

CHAPTER VIII

Problems of the three sections. Other problems whose solutions depend on finding the common points of two co-axial homographic ranges

CHAPTER IX

Involution

CHAPTER X

Involution and harmonic section. Harmonic properties of a quadrangle and quadrilateral. Pole and polar

APPENDIX I

PAPPUS' ACCOUNT OF THE PORISMS OF EUCLID, AND HIS LEMMAS (I—XIX) ON THEM

CHAPTER XI

ANHARMONIC PROPERTIES OF POINTS AND TANGENTS OF A CONIC. THE LOCUS AD TRES ET QUATUOR LINEAS

CHAPTER XII

PASCAL—BRIANCHON—NEWTON—MACLAURIN

CHAPTER XIII

POLE AND POLAR. CONJUGATE POINTS AND LINES. CIRCULAR POINTS AT INFINITY. DESARGUES' THEOREM AND ITS CORRELATIVE. PROPOSITIONS RESPECTING TRIANGLES, QUADRANGLES AND QUADRILATERALS INSCRIBED IN AND CIRCUMSCRIBED ABOUT A CONIC. CONTRA-POLAR CONICS

CHAPTER XIV

Common chords and common tangents of two or more conics (1) passing through four points, (2) touching four straight lines, *i.e.* of pencils and ranges of conics

CHAPTER XV

THE HOMOLOGUE OF A LINE AND CONIC. RELATIONS BETWEEN A PAIR OF COMMON CHORDS AND THE CORRESPONDING PAIR OF TANGENT VERTICES. RELATIONS BETWEEN THE FOUR CONSTANTS OF HOMOLOGY

CHAPTER XVI

Construction of the common chords, tangent vertices, and common self-conjugate triangle of two conics

CHAPTER XVII

Conics having double contact

CHAPTER XVIII

Construction of a conic satisfying certain conditions

CHAPTER XIX

Homographic generalisation of circles and the circular points at infinity, conics and their foci, and other associated points and lines

APPENDIX II

Pascal's Theorem proved for the conic and line-pair by the methods of Euclid and Apollonius

INDEX

CHAPTER I

CROSS-RATIO OF A RANGE OF FOUR POINTS, AND OF A PENCIL OF FOUR LINES

1. THE subject treated in the following pages was very fully investigated by the ancient Greek geometers except as regards its extension to conic sections, where important advances have been made in modern times. The old knowledge has been revived during the last 100 years, chiefly on the Continent, and has been systematised partly by means of taking signs into account in dealing with measurements of lengths and angles, and partly by means of an improved and powerful notation which was rendered possible by this consideration of sign. In the footnotes will be found references (with dates) to the mathematicians to whom these improvements are due, and also to the Greek geometers.

This introduction of sign into the consideration of lines and angles is one of the distinguishing features of the geometry of cross-ratio*.

Any segment of a line is considered to be positive or negative according to the direction in which it is measured, so that, if a and b are two points on a straight line, $ab = -ba$.

* The first work in which we find the principle of signs systematically employed in geometry is Carnot's *Géométrie de position* (1803).

Again, if (A, B) is the angle between two lines which is supposed to be measured by rotation starting from the position A towards that of B, then $(A, B) = -(B, A)$. There is no necessity to do more than allude to this, as the student will have met with a full treatment of it at an earlier stage of his reading.

Range, Axis, Base.

2. DEF. A set of points arranged in any manner on a straight line is called a *range*, and the straight line is called the *axis* or *base* of the range.

Pencil, Centre, Vertex.

A series of concurrent straight lines is called a *pencil*, and their common point is called its *centre* or *vertex*.

The pencils with which we shall deal in the following pages are always coplanar.

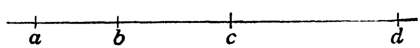

Fig. 1.

Four collinear points taken in pairs give rise to six segments, and these segments have sign as well as magnitude.

Cross-ratio of four points.

An expression such as $\dfrac{ac}{ad} : \dfrac{bc}{bd}$ is called a *cross-ratio* of the four points or range *abcd*, and is the ratio of the distances of the point a from c and d divided by the ratio of the distances of the point b from the same two points, and this ratio of ratios, or cross-ratio, is written $(abcd)$*.

* The term anharmonic function or ratio was given to this expression by Chasles in his *Aperçu historique*, 1837, but the term cross-ratio, *i.e.* ratio of the ratios in which cd is divided by a and b, was introduced by Clifford in 1878, and is now generally adopted. The notation (a, b, c, d) to denote the cross-ratio of four points was employed by Möbius, 1827, but was not adopted by Chasles in his *Géométrie Supérieure*, 1852, although in his *Traité des Sections Coniques*, 1865, he made use of it and of the corresponding notation $P\,(a, b, c, d)$ to denote the cross-ratio of a pencil of four rays. As no advantage is gained by retaining the commas here, we have omitted

3. By varying the order in which the four points are taken we can form 24 cross-ratios, which, however, are not all different in value. For consider

$$(abcd) = \frac{ac}{ad} : \frac{bc}{bd},$$

$$(badc) = \frac{bd}{bc} : \frac{ad}{ac} = \frac{ac}{ad} : \frac{bc}{bd} = (abcd),$$

$$(cdab) = \frac{ca}{cb} : \frac{da}{db} = \frac{ac}{ad} : \frac{bc}{bd} = (abcd),$$

$$(dcba) = \frac{db}{da} : \frac{cb}{ca} = \frac{ac}{ad} : \frac{bc}{bd} = (abcd),$$

i.e. the value of a cross-ratio of four points will be unaltered if we interchange the positions of any pair of points in it, provided we also interchange the positions of the other pair.

It should be noticed that in the above equal cross-ratios the pair of points a and b are associated together, as are also the pair c and d. It does not matter which pair is mentioned first; all that does matter is that if a and b are interchanged, so must also c and d. If we interchange the order of one pair only, we invert the cross-ratio, as shewn below. If we pair the points differently, we get entirely different cross-ratios, as the student may easily see for himself by trial. The relations between them will be given in Art. 4.

Since $(bacd) = \frac{bc}{bd} : \frac{ac}{ad} = \frac{1}{(abcd)}$, *we see that if we interchange separately either the first or last pair of points in a cross-ratio, we shall obtain another cross-ratio which is the reciprocal of the former.*

them, and written the functions simply $(abcd)$ and $P(abcd)$, reserving the commas for use in the notation for involution. If the student uses commas at all, it is best to use one comma only, to separate the pairs, thus (ab, cd).

Consequently, the 24 cross-ratios may be arranged in six groups of four mutually equal cross-ratios, which may be written

$$(abcd),\quad (acdb),\quad (adbc),$$

and

$$(abdc),\quad (acbd),\quad (adcb),$$

of which the first three are formed by the cyclical interchange of the letters b, c, d, and the last three are respectively the reciprocals of the first three.

Relations between the cross-ratios of four given points.

4. There is an important fundamental relation, discovered by Euler in 1747, between the segments of a line made by four points, a, b, c, d, on it, viz.

$$ab \,.\, cd + ac \,.\, db + ad \,.\, bc = 0,$$

in which the second factors of the terms are formed by the cyclical interchange of the letters b, c, d, viz. cd, db, bc, and the first factors are the distances of a from the remaining points. We will prove this and then apply it to find the relations between the various cross-ratios of the four points.

In Fig. 1, if a, b, c, d are the four points

$$cd = ad - ac, \quad db = ab - ad, \quad bc = ac - ab,$$

$$\begin{aligned} \therefore\ & ab \,.\, cd + ac \,.\, db + ad \,.\, bc \\ &= ab\,(ad - ac) + ac\,(ab - ad) + ad\,(ac - ab) \\ &= 0. \end{aligned}$$

Dividing each term by $ad \,.\, bc$ we have

$$-\frac{ab \,.\, cd}{ad \,.\, cb} - \frac{ac \,.\, bd}{ad \,.\, bc} + 1 = 0.$$

$$\therefore\ -\frac{1}{(acdb)} - (abcd) + 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or, if we denote the three cross-ratios $(abcd)$, $(acdb)$, $(adbc)$ by x, y, z, (1) becomes

$$-\frac{1}{y} - x + 1 = 0, \quad \therefore\ \frac{1}{y} = 1 - x, \quad \therefore\ y = \frac{1}{1-x}.$$

Similarly $\dfrac{1}{z} = 1 - y$, and $\dfrac{1}{x} = 1 - z$, $\therefore\ z = \dfrac{x-1}{x}$.

Hence, if the value of one of the cross-ratios is given, we can at once obtain the values of the others. That is, if x is the value of any one of them, the six different values to be found amongst the 24 cross-ratios will be

$$x, \quad \frac{1}{1-x}, \quad \frac{x-1}{x},$$

$$\frac{1}{x}, \quad 1-x, \quad \frac{x}{x-1}.$$

5. A little consideration of the three expressions

$$x, \quad \frac{1}{1-x}, \quad \frac{x-1}{x},$$

will shew the student that two of them are always positive and the third negative, or this may be seen geometrically as follows:

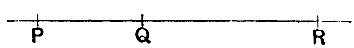

Fig. 2.

Let P, Q, R be three points on a line such that $\dfrac{PR}{PQ} = x$. Then

$$1 - x = -\frac{QR}{PQ}, \quad \text{and} \quad \frac{1}{1-x} = -\frac{PQ}{QR},$$

and

$$\frac{x-1}{x} = 1 - \frac{1}{x} = 1 - \frac{PQ}{PR} = \frac{QR}{PR}.$$

Therefore the six cross-ratios are represented by

$$\frac{PR}{PQ}, \quad \frac{QP}{QR}, \quad \frac{RQ}{RP},$$

$$\frac{PQ}{PR}, \quad \frac{QR}{QP}, \quad \frac{RP}{RQ},$$

where in each line the numerators and denominators of the

second and third fractions are formed by cyclical changes from those of the first fraction.

It is obvious from the figure that two of these six ratios are negative, viz. the two in which the middle letter of the three P, Q, R comes first, whilst the other four are evidently positive.

6. This geometrical connection between the cross-ratios of four collinear points and the simple ratios of the segments formed by three collinear points will probably seem at present artificial, but we shall shortly see that it is an immediate consequence of a very important property of cross-ratios. Meanwhile the student may make use of it to deduce the remaining five cross-ratios of four points when one of them is known.

Ex. 1. If $ab=2''$, $bc=3''$, $cd=1''$, find $(abcd)$, and deduce the values of the other five cross-ratios. *Ans.* $\frac{10}{9}$; -9, $\frac{1}{10}$, $\frac{9}{10}$, $-\frac{1}{9}$, 10.

Ex. 2. Verify Euler's Theorem in the case of the range in Ex. 1.

7. Seeing then that the values of all the different cross-ratios of four collinear points can be expressed in terms of any one of the cross-ratios, it will be found convenient, and it will fix our attention in any investigation if we select any one of the cross-ratios and speak of it as the cross-ratio of the four points that we are considering. It is of course immaterial which of the 24 cross-ratios we take as our standard, and different writers have adopted different orders of the letters. All that is necessary is the observance of consistency. We have invariably adopted the order $\frac{ac}{ad}:\frac{bc}{bd}$, and our reason for doing so is that it is the arrangement adopted by Chasles.

8. *If no two of the four points coincide, a cross-ratio cannot have as its value* 0, $+1$, *or* ∞, *though it is capable of assuming any other value.*

For, taking the cross-ratio to be $\frac{ac}{ad}:\frac{bc}{bd}$, if $\frac{ac\,.\,bd}{ad\,.\,bc}=0$, then either a and c, or b and d coincide.

If $\dfrac{ac \, . \, bd}{ad \, . \, bc} = \infty$, either a and d, or b and c coincide.

If $$\frac{ac \, . \, bd}{ad \, . \, bc} = 1, \quad \frac{ac}{ad} = \frac{bc}{bd} = \frac{ac - ab}{ad - ab},$$

$$\therefore \; ab\,(ad - ac) = 0, \qquad \therefore \; ab \, . \, cd = 0,$$

consequently, either a and b, or c and d coincide.

The relations of Art. 4 shew that the cross-ratios become in pairs 0, $+1$, or ∞, if one of them does, as is also shewn by the geometrical relations of Art. 5.

9. *Given the cross-ratio* (λ) *of four collinear points* a, b, c, d, *of which three are given in position, it is required to find the position of the fourth.*

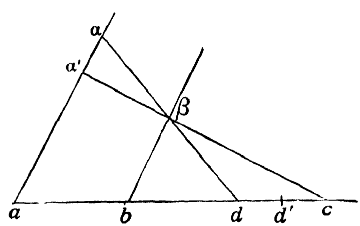

Fig. 3.

Suppose the three given points are a, b, c. Through a draw any straight line $a\alpha$ making any angle with the given line abc, and on it take two points α, α' such that $\dfrac{a\alpha'}{a\alpha} = \lambda$. Join $\alpha' c$. Through b draw $b\beta$ parallel to $a\alpha$, meeting $\alpha' c$ in β. Join $\alpha\beta$ meeting the line abc in d. Then d shall be the point required. For by similar triangles

$$\frac{ac}{bc} = \frac{a\alpha'}{b\beta}, \quad \frac{ad}{bd} = \frac{a\alpha}{b\beta}.$$

Hence $$\frac{ac}{bc} : \frac{ad}{bd} = \frac{a\alpha'}{a\alpha} = \lambda.$$

In the above construction λ is supposed to be positive. If it is negative, we must take α and α' on opposite sides of a.

10. It still remains to be proved that the point d is unique. We have $\frac{ad}{bd} = \frac{1}{\lambda} \cdot \frac{ac}{bc} = \kappa$ suppose, where κ is constant. If possible let d' be another point on the line ab such that $\frac{ad'}{bd'} = \kappa$.

Then
$$\frac{ad'}{bd'} = \kappa = \frac{ad}{bd} = \frac{ad' - dd'}{bd' - dd'}.$$

Therefore either $ad' = bd'$, and consequently $ab = 0$, or else $dd' = 0$. Now a does not coincide with b, therefore d' must coincide with d.

11. Remembering that λ is only one of the cross-ratios of

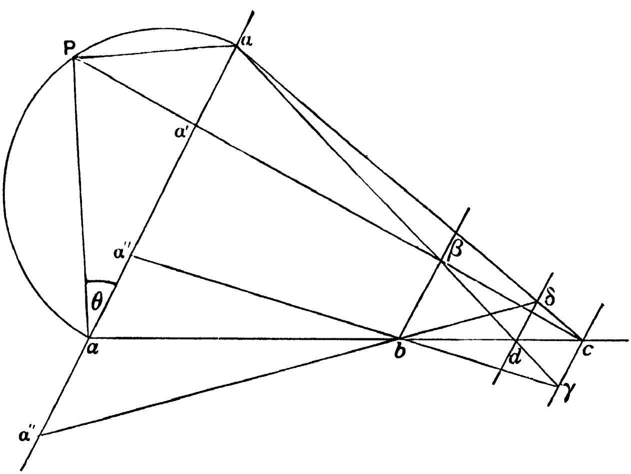

Fig. 4.

the four points a, b, c, d, i.e. $(abcd) = \lambda = \frac{a\alpha'}{a\alpha}$, we can shew on Fig. 4 the values of the other cross-ratios $(acbd)$ and $(adbc)$.

Through c and d draw $c\gamma$, $d\delta$ parallel to $a\alpha$. Let αc meet $d\delta$ in δ, and let αd meet $c\gamma$ in γ. Let $b\gamma$ and $b\delta$ meet $a\alpha$ in α'', α'''.

Then $$(acbd) = \frac{ab}{cb} : \frac{ad}{cd} = \frac{a\alpha''}{c\gamma} : \frac{\alpha a}{c\gamma} = \frac{a\alpha''}{a\alpha},$$

$$(adbc) = \frac{ab}{db} : \frac{ac}{dc} = \frac{a\alpha'''}{d\delta} : \frac{\alpha a}{d\delta} = \frac{a\alpha'''}{a\alpha},$$

so that if $a\alpha$ is taken as the unit of length, $a\alpha'$, $a\alpha''$, and $a\alpha'''$ will represent the values of $(abcd)$, $(acbd)$ and $(adbc)$.

12. Again, describe a semicircle on $a\alpha$ as diameter, draw $\alpha' P$ perpendicular to $a\alpha$, and join Pa, $P\alpha$.

Then $$(abcd) = \frac{a\alpha'}{a\alpha} = \frac{a\alpha'}{aP} \cdot \frac{aP}{a\alpha} = \cos^2 \theta.$$

Therefore by Art. 4, the six cross-ratios of any four collinear points can be represented by

$$\begin{array}{lll} \cos^2 \theta, & \operatorname{cosec}^2 \theta, & -\tan^2 \theta, \\ \sec^2 \theta, & \sin^2 \theta, & -\cot^2 \theta. \end{array}$$

This also follows from the fractions of Art. 5. For if we take three collinear points in the order PQR as in Fig. 2, and draw a semicircle on PR as diameter, and draw the perpendicular QS cutting the semicircle in S, and join SP, then if the angle $SPR = \theta$, the values of the six ratios are $\cos^2 \theta$ &c. as above.

If we take the angle ϕ at R we shall obtain the same results but in different order, as the two angles θ and ϕ are complementary.

It will be noted that the negative ratios are $-\tan^2\theta$ and $-\cot^2\theta$, of which one is > -1, and the other < -1, while of the four positive ratios two must be > 1 and two < 1.

The student should also notice that the points P, Q, R are the same as the points a, α', α of Art. 11 and Fig. 4.

The method of Art. 5 is most suitable for expressing the six cross-ratios of four points as vulgar fractions, whilst by Art. 11 or 12 (see Art. 13 below), we can most easily express them as decimals.

13. Suppose now the range *abcd* is given, and we wish to find geometrically the value of its cross-ratio.

In Fig. 3, through *a* and *b* draw the parallels $a\alpha$, $b\beta$, and on the first take $a\alpha$ of unit length. Join αd cutting $b\beta$ in β. Join $c\beta$, cutting $a\alpha$ in α'. Then $a\alpha'$ is the cross-ratio required.

14. Def. If we consider the four concurrent straight lines OA, OB, OC, OD, we define the compound ratio $\dfrac{\sin(A, C)}{\sin(A, D)} : \dfrac{\sin(B, C)}{\sin(B, D)}$ formed by taking the sines of four of the six angles which these lines make with one another as the *cross-ratio of the pencil $O(ABCD)$*. See also Art. 16.

Cross-ratio of four lines.

15. *If the pencil $O(ABCD)$ is cut by a transversal in the four points a, b, c, d, the cross-ratio of the pencil will be equal to that of the range abcd, and will have the same sign.*

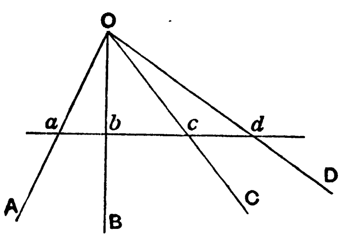

Fig. 5.

For $$\frac{\sin(A, C)}{\sin c} = \frac{ac}{aO}, \qquad \frac{\sin(A, D)}{\sin d} = \frac{ad}{aO},$$

$$\therefore \quad \frac{\sin(A, C)}{\sin(A, D)} = \frac{\sin c}{\sin d} \cdot \frac{ac}{ad}.$$

Similarly $$\frac{\sin(B, C)}{\sin(B, D)} = \frac{\sin c}{\sin d} \cdot \frac{bc}{bd}.$$

$$\therefore \quad \frac{\sin(A, C)}{\sin(A, D)} : \frac{\sin(B, C)}{\sin(B, D)} = \frac{ac}{ad} : \frac{bc}{bd}.$$

Now if we suppose the usual convention to hold respecting the positive and negative directions of rotation in the description of angles, so that $\sin(A, C) = -\sin(C, A)$, &c., it is clear that *the cross-ratio of the pencil $O(ABCD)$ is equal to the cross-ratio of the range abcd, both in magnitude and in sign.*

16. We may remark that the definition given in Art. 14 is not often used, and in view of the property proved in Art. 15 it would be better to substitute the following:

DEF. The cross-ratio of a pencil of four rays is that of the range which it forms on any transversal.

As in Art. 2, when we refer to the cross-ratio of the pencil $O(ABCD)$, we shall speak of it either as the pencil $O(ABCD)$, or simply $O(ABCD)$.

The conclusions of Arts. 3—8 respecting the cross-ratios of four points will apply equally to the cross-ratios of a pencil of four rays, and need not be repeated.

17. *Given λ, the cross-ratio of four rays OA, OB, OC, OD of a pencil of which the first three rays OA, OB, OC are given in position, it is required to find the fourth ray.*

In Fig. 6 let any transversal meet the three given rays in a, b, c.

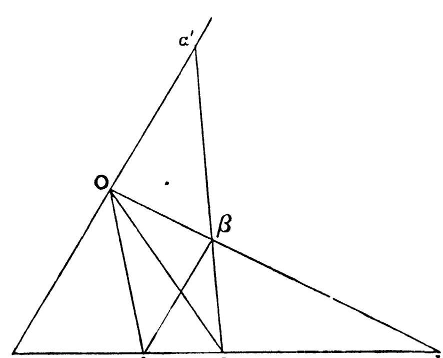

FIG. 6.

On aO take the point a' so that $\frac{aa'}{aO} = \lambda$. Join $a'c$, and through b draw $b\beta$ parallel to aO, cutting $a'c$ in β. Join $O\beta$. This is the fourth ray required.

Produce $O\beta$ to cut abc in d.

Then $$\frac{ac}{ad} : \frac{bc}{bd} = \frac{ac}{bc} : \frac{ad}{bd} = \frac{aa'}{b\beta} : \frac{aO}{b\beta} = \frac{aa'}{aO} = \lambda.$$

It should be noted that the above construction is the same as that given in Art. 9.

18. MENELAUS' THEOREM. *If any transversal cuts the sides of a triangle ABC in the points a, b, c,*

$$Ab \, . \, Ca \, . \, Bc = - bC \, . \, aB \, . \, cA.$$

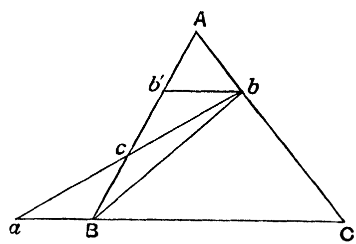

Fig. 7.

Through b draw bb' parallel to BC, and join bB.

Then by Art. 15, the range $(cb'AB)$ = the pencil $b\,(cb'AB)$
= the range $(a \infty CB)$.

Therefore $$\frac{cA}{cB} : \frac{b'A}{b'B} = \frac{aC}{aB} : \frac{\infty C}{\infty B}.$$

Now $$\frac{b'A}{b'B} = \frac{bA}{bC}, \quad \text{and} \quad \frac{\infty C}{\infty B} = 1.$$

$$\therefore \quad \frac{cA}{cB} : \frac{bA}{bC} = \frac{aC}{aB}.$$

Hence $$Ab \, . \, Ca \, . \, Bc = - bC \, . \, aB \, . \, cA.$$

Conversely, if three points a, b, c on the sides of a triangle ABC satisfy the relation $Ab \,.\, Ca \,.\, Bc = -bC \,.\, aB \,.\, cA$, they are collinear. For suppose that bc when produced meets BC in a'. Then by the above,

$$Ab \,.\, Ca' \,.\, Bc = -bC \,.\, a'B \,.\, cA,$$

and by hypothesis

$$Ab \,.\, Ca \,.\, Bc = -bC \,.\, aB \,.\, cA.$$

Therefore $$Ca' : Ca = a'B : aB,$$

$$Ca' : a'B = Ca : aB,$$

$$CB : a'B = CB : aB.$$

Consequently the point a' coincides with a.

NOTE. Menelaus was a Greek geometer and astronomer, a native of Alexandria. He was at Rome studying astronomy in the first year of Trajan, A.D. 93. The theorem in the text is given in his treatise on Spherical Trigonometry in 3 books, which survived in Arabic, and of which a Latin translation was first published in a Collection of Greek Geometers made at Paris in 1626.

EXAMPLES.

1. Any two transversals cut the sides of a triangle in the points P, Q, R and P', Q', R'. Prove that

$$(BCPP')(CAQQ')(ABRR') = 1.$$

Expand and use Menelaus' Theorem.

2. If a transversal meets the consecutive sides of a polygon $ABCD\ldots$ in the points a, b, c ..., shew that

$$aA \,.\, bB \,.\, cC \ldots = aB \,.\, bC \,.\, cD \ldots.$$

3. Shew that (1) $(PQRT) \times (PQTS) = (PQRS)$.

(2) $(PTRS) \times (TQRS) = (PQRS)$.

[The student should notice the position of the element T in the factors.]

4. If one of the cross-ratios of a range $= -1$, find the values of the other cross-ratios.

CHAPTER II

EQUICROSS RANGES AND EQUICROSS PENCILS. PERSPECTIVE

19. *Given a range of four points abcd on one straight line, and a range of four points a′b′c′d′ respectively corresponding to them on another straight line, if a cross-ratio of the first range is equal to the corresponding cross-ratio of the second, then the other cross-ratios of the first range are respectively equal to the corresponding cross-ratios of the second.*

For suppose $$(abcd) = (a'b'c'd').$$

Then by Art. 4 (1),

$$(abcd) = 1 - \frac{1}{(acdb)},$$

and $$(a'b'c'd') = 1 - \frac{1}{(a'c'd'b')}.$$

Therefore $$(acdb) = (a'c'd'b').$$

Similarly $$(adbc) = (a'd'b'c').$$

20. It follows at once from Art. 4 that if two ranges of four points have a cross-ratio of the one equal to a cross-ratio of the other, then each one of the 24 cross-ratios of the one is equal to the corresponding cross-ratio of the other; and consequently we may briefly say that two such ranges have their cross-ratios equal, or have the same cross-ratios, or we may speak of them still more briefly as equicross, or even as equal ranges.

If we have two ranges whose corresponding segments are proportional, we shall call them similar ranges, and if the corresponding segments are equal, the ranges are said to be identical, and are then superposable. All such ranges are, of course, equicross.

Ex. In the range $(abcd)$ $ab=3$ cm., $bc=2$ cm., $cd=1$ cm. In an equicross range $(a'b'c'd')$ find the position of d' when

(1) $a'b'=2$ cm., $b'c'=3$ cm. (2) $a'b'=5$ cm., $b'c'=6$ cm.

(3) $a'b'=4$ cm., $b'c'=3$ cm. (4) $a'b'=6$ cm., $b'c'=4$ cm.

Ans. (1) $c'd'=3$; (2) $c'd'=4\frac{5}{7}$; (3) $c'd'=1\frac{3}{13}$; (4) $c'd'=2$.

It should be noticed that ranges which are equicross are not usually similarly divided, though of course they may be so. Thus comparing the range $(abcd)$ with the equal ranges (1), (2), (3), (4), we see that only in the last are the segments proportional to those of $(abcd)$.

Ranges in Perspective.

21. If a pencil is cut by two transversals in the points $abcd$, $a'b'c'd'$, the ranges are equicross by Art. 15, for each of them is equal to the cross-ratio of the pencil. Hence

A pencil cuts any two transversals in equicross ranges.*

This is the fundamental proposition of the subject, and is a projective property. It is important to notice that while the ratio of the segments into which a transversal is divided by a pencil of *three* rays is the same only for parallel transversals and for a pencil of parallel rays, the cross-ratio for a pencil of *four* rays is the same for all transversals.

Perspective, Centre of Perspective.

Def. When two equicross ranges are so placed that they are transversals of the same pencil, *i.e.* when the lines joining corresponding points on the ranges are concurrent, the ranges are said to be in *perspective*, and the vertex of the pencil is called the *centre of perspective*.

* Pappus, Bk vii, Prop. 129.

If the ranges are in perspective, they are necessarily equicross, but the converse, as we shall see later, is by no means necessarily true. What happens when they are not in perspective is discussed in Chapter XI. We can, however, always place equicross ranges so that they shall be in perspective; a simple method of doing this is given in Art. 23.

If the two ranges when in perspective are parallel it is easy to see that they are similar, and conversely, to put similar ranges in perspective all that is needed is to make them parallel.

If two ranges are *identical* they will also be in perspective when they are parallel, the centre of perspective being at infinity.

All this was well known to and discussed by the ancient Geometers, as the student will see by referring to Appendix I.

22. *Given two equicross ranges, if the lines joining 3 pairs of corresponding points pass through a point, the line joining the fourth pair will also pass through the same point.*

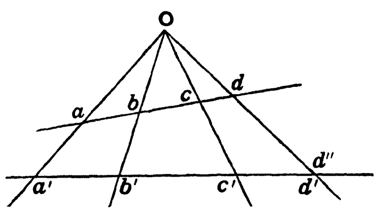

Fig. 8.

Let $(abcd)$ and $(a'b'c'd')$ be two equicross ranges, and let aa', bb', cc' meet in O. If Od does not pass through d', let it meet the line $a'b'c'$ in d''.

Then $\qquad (a'b'c'd'') = (abcd)$ by Art. 21

$\qquad\qquad\qquad = (a'b'c'd')$ by hypothesis.

Therefore, by Art. 10, d'' coincides with d'.

23. Def. In two equicross ranges, if a point of one coincides with the corresponding point of the other, the two ranges are said to have a *common point*.

Common point.

• *If two equicross ranges have a common point, then the straight lines joining the other pairs of corresponding points are concurrent.*

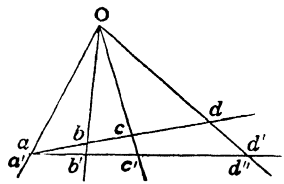

Fig. 9.

Let the points a, a' coincide, and let bb', cc' meet in O.
Join Oa, and let Od meet $a'b'c'$ in d''.

Then $$(abcd) = (a'b'c'd'') \text{ by Art. 21}$$
$$= (a'b'c'd') \text{ by hypothesis.}$$

Therefore, by Art. 10, d'' coincides with d'.

Note. When this is the case the ranges are in *perspective*, centre O, but of course they may be in perspective even if (a, a') do not coincide, but it is not so easy to put them in perspective in that case. This will be discussed when we come to homographic ranges in which there are more than 4 points on each range.

Pencils in Perspective.

24. *If two pencils are subtended by the same range, they are equicross.*

By Art. 15 the cross-ratios of each of the pencils are equal to the cross-ratios of the range $abcd$.

Def. When two equicross pencils are so placed that they subtend the same range, *i.e.* so that corresponding rays intersect in collinear points, the pencils are said to be in *perspective*, and the common range

Perspective, Axis of Perspective.

is called the *axis of perspective*. A simple method of putting them in perspective is given in Art. 25.

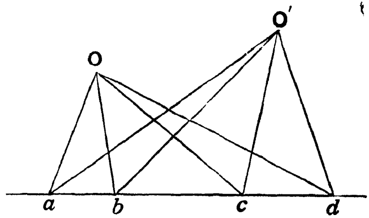

Fig. 10.

Just as in the case of ranges, if two pencils are equicross, they are not necessarily in perspective, and what happens in general is discussed in a later chapter.

25. DEF. In two equicross pencils if a ray of one pencil coincides with the corresponding ray in the other, the two pencils are said to have a *common ray*.

Common Ray.

If two equicross pencils of 4 *corresponding rays have a* common *ray, then the other pairs of corresponding rays will intersect in three points which are collinear, and conversely.*

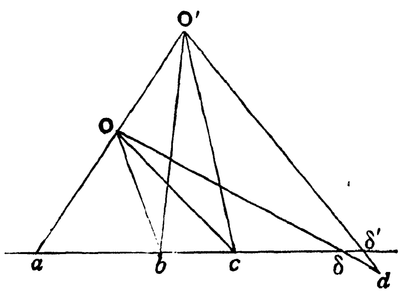

Fig. 11.

Let the rays of the pencils be Oa, Ob, Oc, Od, and $O'a$, $O'b$, $O'c$, $O'd$, so that the rays Oa, $O'a$ coincide. Join bc meeting the *common* ray in a, and let bc meet the rays Od, $O'd$ in δ, δ'. Then

since the two pencils are equicross, $(abc\delta) = (abc\delta')$, and therefore by Art. 10 the points δ, δ' coincide, *i.e.* the line bc passes through d.

Conversely, if two pencils are such that the intersections of three pairs of corresponding rays are collinear, and the fourth pair of rays are in the same straight line, the pencils are equicross.

For in Fig. 11 suppose the point d to lie on the line bc. Then by Art. 24 the pencils $O(abcd)$ and $O'(abcd)$ are equicross.

If two equicross pencils with centres O, O', are such that the intersections of three pairs of corresponding rays are collinear, as a, b, c in Fig. 10, *then the fourth pair of rays will also intersect on the line abc*, for if they meet this line in two separate points δ, δ', we should have $(abc\delta) = (abc\delta')$, and therefore, by Art. 10, δ and δ' must coincide.

In all the above cases the pencils are in *perspective*, and the line on which the pairs of corresponding rays intersect is *the axis of perspective.*

Triangles in Perspective.

26. DEF. If two triangles are such that the lines joining the pairs of corresponding vertices are concurrent, the triangles are said to be *co-polar*.

Co-polar Triangles.

If two triangles are such that the intersections of pairs of corresponding sides are collinear, the triangles are said to be *co-axial.*

Co-axial Triangles.

Two triangles which are co-polar are also co-axial, and two triangles which are co-axial are also co-polar.

Let abc, $a'b'c'$ be two triangles such that the lines aa', bb', cc' meet in the point O. Let $(bc, b'c')$ meet in α, $(ca, c'a')$ in β and $(ab, a'b')$ in γ. It is required to prove that α, β, γ are collinear.

Let the line Occ' meet ab in δ and $a'b'$ in δ'.

Then by Art. 21 the range $(a\delta b\gamma)$ = the range $(a'\delta'b'\gamma)$, and by Art. 15 the pencil $c(a\delta b\gamma)$ = the pencil $c'(a'\delta'b'\gamma)$.

Now the corresponding rays $c\delta$ and $c'\delta'$ are also *common* rays,

and therefore by Art. 25 the intersections of $(ca, c'a')$, $(cb, c'b')$, $(c\gamma, c'\gamma)$ are collinear, *i.e.* the points β, α, γ are collinear.

Conversely, let the points α, β, γ be collinear. It is required to shew that aa', bb', cc' are concurrent.

Since the pencils $c(a\delta b\gamma)$ and $c'(a'\delta' b'\gamma)$ are such that the intersections of three pairs of corresponding rays are collinear, and the fourth pair of rays are common, the pencils are equicross by Art. 25.

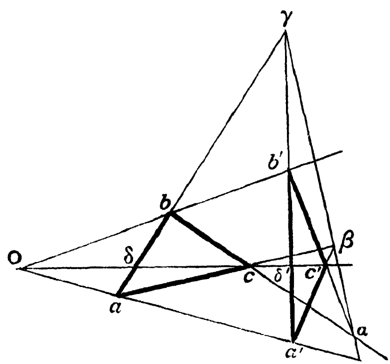

Fig. 12.

Therefore $\qquad c(a\delta b\gamma) = c'(a'\delta' b'\gamma)$,

$\therefore$ by Art. 15 $\qquad (a\delta b\gamma) = (a'\delta' b'\gamma)$,

Perspective, Pole, Centre of Perspective, Axis of Perspective.

and by Art. 23, aa', $\delta\delta'$, bb' are concurrent.

Two triangles which have the above properties are said to be in *perspective*. The point O is called the *pole*, or *centre of perspective*, and the line $\alpha\beta\gamma$ the *axis of perspective*.

Ex. 1. In Fig. 8. If the ranges intersect in e shew that the range formed from any four of the points a, b, c, d, e is equicross with the corresponding range on the other transversal.

Ex. 2. If two pencils $O(abcd)$, $O'(abcd)$ are in perspective shew that the pencil formed from any four of the rays $O(O'abcd)$ is equicross with the corresponding pencil from O'.

CHAPTER III

HARMONIC RATIO

Harmonic Range, Harmonic Conjugates.

27. DEF. If the points a, a' divide a segment ef internally and externally in a given ratio, *i.e.* in such a way that the cross-ratio $(aa'ef)$, *i.e.* $\frac{ae}{af} : \frac{a'e}{a'f} = -1$, the four are said to form a *harmonic range*; the points a, a' are said to divide the segment ef harmonically, and are called the *harmonic conjugates* of the points e, f.

Since the above relation may be written $\frac{ea}{ea'} : \frac{fa}{fa'} = -1$, we see also that the points e, f divide the segment aa' harmonically, and that they are the harmonic conjugates of the points a, a'.

Fourth Harmonic, Harmonic Conjugate.

The points a, a' may be spoken of as a pair of conjugate points, and similarly for e, f. Also each point of a pair of conjugates is called the *fourth harmonic*, or the *harmonic conjugate* of the other for the second pair of points.

28. Substituting the value -1 in any one of the six expressions in Art. 4 we see that when a range is harmonic the six values of the cross-ratios reduce to three, viz. -1, $\frac{1}{2}$, 2.

Conversely, if a range has one of its cross-ratios equal to either -1, or $\frac{1}{2}$, or 2, the range is harmonic.

Since -1 is the value of the cross-ratio $(aa'ef)$, where aa' and ef are pairs of conjugate points, the reciprocal of this ratio, *i.e.* $(aa'fe)$, see Art. 3, must also $= -1$.

Hence $(aa'ef) = (aa'fe) = (a'aef)$ by Art. 3. Consequently, *a range is harmonic if it has a cross-ratio whose value is unaltered on interchanging the positions of a single pair of points. When this is the case, the pair of points so interchanged are conjugates.*

We may remark that not only is $(aa'ef) = (a'aef)$, but it follows that

$$(aea'f) = (a'eaf) = (afa'e),$$

and

$$(aefa') = (a'efa) = (afea'),$$

i.e. in a harmonic range each of either pair of conjugate points is interchangeable in any of the cross-ratios, not only in the fundamental cross-ratio whose value is -1.

If we take the other values of the cross-ratios, viz. $(aea'f) = 2$, we have

$$\frac{aa'}{af} : \frac{ea'}{ef} = 2, \quad \therefore \; aa' . ef = 2af . ea',$$

and $\quad (afea') = \frac{1}{2}, \quad \dfrac{ae}{aa'} : \dfrac{fe}{fa'} = \frac{1}{2}, \quad \therefore \; aa' . ef = 2ae . a'f.$

These results can also be easily obtained from Euler's Equation, Art. 4,

$$aa' . ef + ae . fa' + af . a'e = 0.$$

For since $\quad (aa'ef) = -1, \quad$ *i.e.* $\dfrac{ae . a'f}{af . a'e} = -1,$

therefore $\quad aa' . ef = 2ae . a'f = -2af . a'e.$

Ex. If $aa' = 4$ cm. and is divided by e and f internally and externally in the ratio $3:1$, find the values of $aa' . ef$, $ae . a'f$ and $af . a'e$, and verify the above relations.

29. *To find the fourth harmonic of three given points.*

We will employ the method of Art. 9. See also Art. 32, Cor.

(1) Let the segment aa' be cut internally at the point e. It is required to find the fourth harmonic of e for the points a, a'.

Through a and a' draw any two parallel straight lines $a\alpha$, $a'\beta$, and on $a\alpha$ take two points α, α' such that $a\alpha = -a\alpha'$.

Draw αe meeting $a'\beta$ in β. Join $\alpha'\beta$ meeting aa' in f. Then f will be the point required.

For
$$\begin{aligned} ae : ea' &= \alpha a : \beta a' \\ &= \alpha' a : \beta a' \\ &= af : a'f. \end{aligned}$$

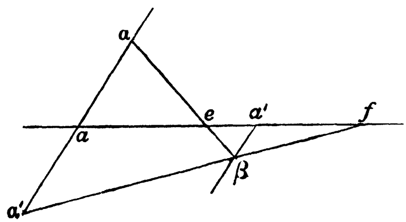

Fig. 13.

(2) Suppose the segment aa' is cut externally at f. As before, take $a\alpha = -a\alpha'$. Join $\alpha' f$ cutting the parallel $a'\beta$ in β. Then $\alpha\beta$ will meet aa' in the required point e.

Cor. If one of the pair of conjugate points ef is at infinity, as f suppose, then e is the mid-point of aa', as is obvious from the construction in Fig. 13, since $\alpha' f$ will then be parallel to aa', and $a'\beta$ will $= a\alpha' = -a\alpha$.

This also follows from the algebraical relation $(aa'ef) = -1$, as the student should verify, noting that $\dfrac{af}{a'f} = 1$, and therefore $\dfrac{ae}{a'e} = -1$. Hence

If one of the four points of a harmonic range is at infinity, its conjugate is at the mid-point between the other two, and vice versa.

30. *Given three rays of a pencil, to find the fourth harmonic of one of them for the other two.*

Let PA, PB be two of the given rays, PC the third. It is required to find the fourth harmonic of PC for PA and PB.

Draw any line ab parallel to PC meeting PA, PB or these lines produced in a, b, and bisect ab in d. Then Pd is the ray required.

For since ab is bisected at d, the range $(ab \infty d)$ is harmonic,

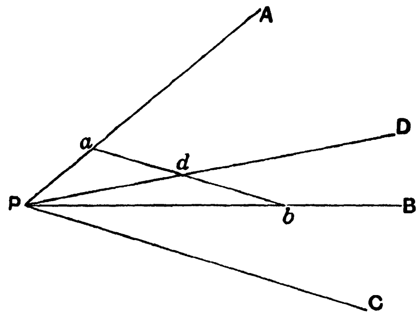

Fig. 14.

by Art. 29, and consequently by Art. 15 the pencil $P(ABCD)$ is harmonic.

Cor. From the above construction it is evident that if one of the rays PC, PD bisects the interior angle between PA and PB, its conjugate will bisect the exterior angle between them. Hence,

If one of three rays of a pencil bisects the angle between the other two, its conjugate is at right angles to it.

31. *Every harmonic range determines a harmonic pencil at every centre, and every harmonic pencil determines a harmonic range on every transversal. See Arts.* 15 *and* 21.

Relations between the Segments of a Harmonic Range.

32. Let aa', ef be two pairs of conjugate points forming a harmonic range, and let O, O' be the mid-points of aa' and ef respectively.

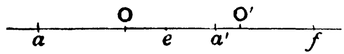

Fig. 15.

Then $$Oe \, . \, Of = Oa^2 = Oa'^2.$$

For since $$\frac{ae}{af} + \frac{a'e}{a'f} = 0 \, *,$$

$$\therefore \quad \frac{Oe - Oa}{Of - Oa} = -\frac{Oe - Oa'}{Of - Oa'} = -\frac{Oe + Oa}{Of + Oa};$$

whence, clearing of fractions, we have at once

$$2Oe \, . \, Of = 2Oa^2,$$

the other terms cancelling.

$$\therefore \quad Oe \, . \, Of = Oa^2 = Oa'^2 = (\tfrac{1}{2} aa')^2 \, *.$$

Similarly $$O'a \, . \, O'a' = (\tfrac{1}{2} ef)^2 \, \dagger.$$

COR. This gives rise to another construction for the fourth harmonic.

On aa' as diameter describe a circle, and let $\alpha\alpha'$ be the diameter perpendicular to aa'. Let αe meet the circle in P. Then $\alpha' P$ will meet aa' in the required point f. This is obvious from consideration of similar triangles.

33. Another interesting property is that

$$\frac{1}{ae} + \frac{1}{af} = \frac{2}{aa'}.$$

For since $$\frac{ae}{af} + \frac{a'e}{a'f} = 0,$$

* Pappus, Bk VII, Lemma XXXIV.

† Pappus, Bk VII, Lemmas XXVI, XXVII.

$$\therefore \quad \frac{ae}{af} + \frac{ae - aa'}{af - aa'} = 0,$$

$$\therefore \quad aa' . af + aa' . ae = 2ae . af,$$

$$\therefore \quad \frac{1}{ae} + \frac{1}{af} = \frac{2}{aa'}.$$

This property shews that aa' is the harmonic mean between the segments ae, af, and may be considered the reason for the name 'harmonic' being applied to the range.

34. The following 7 relations are given by Pappus, Bk VII, in his Lemmas to Euclid's Porisms, and are left as exercises to the student.

(1) $OO'^2 = aO^2 + eO'^2$ Lemmas xxii and xxiv.

(2) $ae^2 = 2aO' . Oe$ " xxiii and xxv.

(3) $ea'^2 = 2a'O' . Oe$ " "

(4) $\dfrac{ae^2}{ea'^2} = \dfrac{aO'}{a'O'}$ " xxvi and xxvii.

(5) $ae . ea' = Oe . ef$

(6) $af . a'f = Of . ef$

(7) $\dfrac{ae}{ea'} = \dfrac{af}{a'f}$

(5)–(7): Lemma xxxiv.

(8) Shew that if the cross-ratio $(aa'ef)$ is equal to either $-\frac{1}{2}(aea'f)$ or $-2(afea')$ the range $aa'ef$ is harmonic, aa' and ef being pairs of conjugate points.

35. *Given two segments aa', bb' on a line, it is required to find on the same line a point O such that $Oa . Oa' = Ob . Ob'$.*

Assuming the existence of such a point, we have

$$Oa . Oa' = Ob . Ob',$$

$$\therefore \quad \frac{Oa}{Ob} = \frac{Ob'}{Oa'} = \frac{Ob' - Oa}{Oa' - Ob} = \frac{ab'}{ba'}.$$

Hence the construction. Through a and b draw two parallel lines, and on them take $a\alpha = ab'$, and $b\beta = ba'$. Then the point where $\alpha\beta$ meets the given line is the required point O.

If the segments aa', bb' do not overlap, the products $Oa \,.\, Oa'$ and $Ob \,.\, Ob'$ are both positive, and we may put them $= Oe^2$ or Of^2. Then each of the segments aa', bb' will be divided harmonically by the real points e, f.

If the segments aa', bb' overlap, the products $Oa \,.\, Oa'$ and $Ob \,.\, Ob'$ are both negative, and consequently the points e, f are imaginary, but we shall still say that the segments aa', bb' are divided harmonically by the imaginary points e, f.

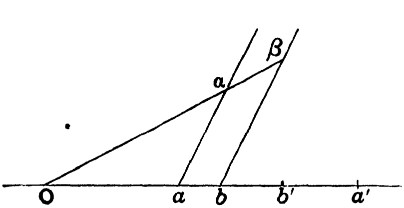

Fig. 16.

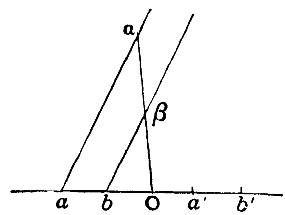

Fig. 17.

It is evident from the construction that for any given position of the segments aa', bb' there is one and only one position of the point O, and only one position of the segment ef. Hence

Only one segment can be found to divide two given collinear segments harmonically.

36. Given a pencil of four rays, if we take the rays in two pairs in any manner, then in each arrangement we can always find a third pair of rays which will form a harmonic pencil with each of the given pairs.

This follows at once from the preceding Art. by drawing a transversal cutting the arranged pairs in aa', bb', and finding the harmonic conjugates e, f of their segments. Then if O is the vertex of the given pencil, Oe, Of are conjugates for Oa, Oa', and also for Ob, Ob'. The rays Oe, Of will be real or imaginary according as we take the two given pairs to be non-overlapping or otherwise.

CHAPTER IV

HOMOGRAPHIC RANGES AND HOMOGRAPHIC PENCILS

37. WHEN two ranges are in perspective we can have as many points as we please on each by supposing a ray to rotate round the centre of perspective, and so determine a moving point on each range such that the two moving points are always a pair of corresponding points. The property of such ranges in perspective is that the cross-ratio of *any* four points on one is equal to the cross-ratio of the corresponding four points on the other. Now suppose that in this way a whole series of points are fixed on the two ranges, and then the ranges are moved away so as no longer to be in perspective, without disturbing the relative positions of the points on each range*. The quality of cross-ratios of course still remains. Such ranges are called homographic. In the next article the definition is given more formally, and without reference to perspective properties.

Homographic Ranges.

38. DEF. If two straight lines are divided at corresponding points in such a manner that the cross-ratio of any four points of the one is equal to the cross-ratio of the four corresponding points of the other, the two straight lines are said to be divided *homographically*, and their points of division are said to form two *homographic ranges*.

* We may even move the ranges so that these lines coincide, having of course the ranges distinctly marked to prevent confusion. This case is very important, and is discussed later.

It is important to emphasize the fact that when we speak of two lines L, L' being divided homographically, we mean that every point on L belongs to the first range and every point on L' belongs to the second range, and that each point on either of the lines corresponds to one and only one point on the other. This correspondence may be arranged in an infinite number of ways by taking any three points a, b, c on L, and any three a', b', c' on L' as their correspondents. Then, corresponding to any position of a variable point m on L, we can find one, and only one point m' on L' such that $(abcm) = (a'b'c'm')$, for the value of the cross-ratio is then fixed, and by Art. 10 the point m' is unique. Of course, for every different position of m the cross-ratio $(abcm)$ will have a different value, but the points a, b, c and a', b', c' will remain unchanged, and will, as it were, determine the character of the different cross-ratios. For this reason we shall often refer to the sets of points abc and $a'b'c'$ as the *characteristics* of the ranges L and L'.

Characteristic of a Range.

For shortness we shall often denote a range $(abcde \ldots)$ by (a), and a pencil $P(abcde \ldots)$ by $P(a)$.

39. *Ranges which are homographic to the same range are homographic to one another.*

From Def. Art. 38, it follows at once that if a range of points (a) is homographic to a range (a'), and also to a range (a''), then the cross-ratio of any four points of the range (a') will be equal to the cross-ratio of the four corresponding points of the range (a''), and therefore by Art. 38 the ranges (a') and (a'') are homographic.

40. *Given two lines L, L', one of which, L, is divided in any manner at the points a, b, c, d ..., it is required to find on L' corresponding points a', b', c', d' ... so that the two lines may be divided homographically.*

As pointed out in Art. 38 this may be done in an infinite number of ways, because we may select any three points a', b', c'

that we please on L', and take them as the points corresponding to a, b, c on L, we can then by Art. 9 find the points d', e'... corresponding to d, e....

It should be noticed that there is one range, and one only, corresponding to each selection of $a'b'c'$.

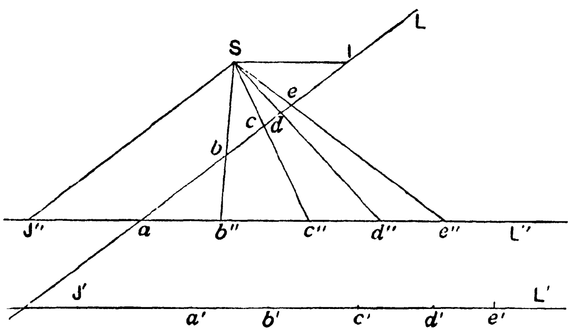

Fig. 18.

One practical way of finding the series of points on L' after having determined on the three characteristic points $a'b'c'$ is as follows:

Through a draw a line L'' making any convenient angle with L, and take $ab'' = a'b'$, $ac'' = a'c'$, and let bb'', cc'' meet in S. Then the lines drawn from S to the points d, e... will meet L'' in points d'', e''..., and if on L' we take $a'd' = ad''$, $a'e' = ae''$..., the points d', e'... will be the points required on L' corresponding to the points d, e ... on L.

That is, the line L'' and the range on it are only the line L' and its range moved to a position in which it is in perspective with the range on L. See Art. 37.

In particular we can find the points on each which correspond to points at infinity on the other by drawing through S lines parallel to L'' and L, meeting L, L'' in I and J''. The point I

will correspond to the point at infinity on L'', and therefore also to the point at infinity on L'. The point J'' will correspond to the point at infinity on L, and if on L' we take $a'J' = aJ''$, the point J' will correspond to the point at infinity on L. See also Art. 46, Fig. 20, and Art. 54.

This construction can be applied to the case where the points a', b', c' are on the line L, *i.e.* when the line L' coincides with the line L, which is sometimes desirable, and which has been referred to in the footnote to Art. 37.

41. The following results are important and obvious extensions of the theorems in Arts. 21, 23.

Common Point of Two Ranges.

(1) *If two straight lines aL, aL'' are cut by a pencil, they are divided homographically.* The point a evidently represents two coincident corresponding points, and is said to be a *common point* of the two ranges on L, L''.

(2) *If two straight lines are divided homographically, and if their point of intersection is a* common point *of the ranges, the straight lines joining the other pairs of corresponding points are all concurrent, and the ranges are said to be* in perspective.

42. DEF. When two pencils, each containing any number of rays, are such that they have each ray of one pencil corresponding to a ray of the other in such a way that the cross-ratio of any four rays of the one is equal to the cross-ratio of the four corresponding rays of the other, the pencils are said to be *homographic*.

Homographic Pencils.

A similar remark to that made in Art. 38 respecting homographic ranges might be made here respecting homographic pencils.

By Art. 15 if a pencil is cut by a transversal, it will be permissible and convenient to say that the pencil and transversal are homographic.

Superposable or Identical Pencils.

DEF. If two pencils having either the same or different vertices are such that the angles between each pair of rays of the one are equal to the angles taken in the same sense between the corresponding pairs of rays of the other, the pencils are said to be *superposable* or *identical.*

Superposable pencils are obviously homographic.

43. *If two pencils $O(ABC\ldots)$, $O'(A'B'C'\ldots)$ are homographic, and from O rays are drawn perpendicular to the rays $O'A'$, $O'B'\ldots$ and forming the pencil $O(abc\ldots)$, and if the pencil $O'(a'b'c'\ldots)$ is formed from $O(ABC\ldots)$ in a similar manner, the pencils $O(abc\ldots)$ and $O'(a'b'c'\ldots)$ are homographic.*

For the angle $O'Oa$ is the complement of $OO'A'$, and $O'Ob$ is the complement of $OO'B'$, therefore $aOb = A'O'B'$, &c. Therefore the pencils $O(abc\ldots)$, $O'(A'B'C'\ldots)$ are superposable. Similarly the pencils $O'(a'b'c'\ldots)$, $O(ABC\ldots)$ are superposable. But $O(ABC\ldots)$ and $O'(A'B'C'\ldots)$ are homographic, therefore so also are $O(abc\ldots)$ and $O'(a'b'c'\ldots)$.

44. *Pencils which are homographic to the same pencil are homographic to one another.*

From Def. Art. 42 it follows that if a pencil $P(a)$ is homographic to a pencil $P'(a')$, and also to a pencil $P''(a'')$, then the cross-ratio of any four rays of the pencil $P'(a')$ will be equal to the cross-ratio of the four corresponding rays of the pencil $P''(a'')$, and therefore the pencils $P'(a')$ and $P''(a'')$ are homographic.

45. *If two pencils are drawn from two centres O, O', and are such that their rays intersect by pairs in a series of collinear points, the pencils are homographic.*

For each of the pencils is homographic with the range on their common transversal abc, Fig. 19.

If the line OO' be produced to meet the transversal abc in K, the line OK will evidently represent two coincident corresponding rays, and may be called a *common ray* of the two pencils.

Common Ray of Two Pencils.

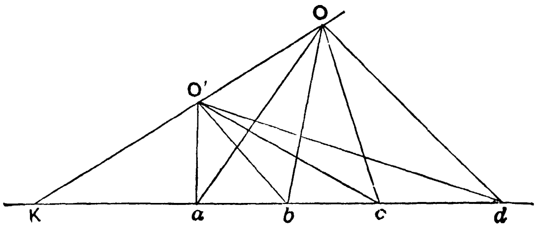

Fig. 19.

Pencils in Perspective.

Conversely, if two homographic pencils have a common ray, *their other pairs of corresponding rays will intersect in a series of points which are collinear, and the pencils are said to be* in perspective. Art. 25.

46. *To find a range which will be in perspective with each of two given homographic ranges which are not in perspective with each other.*

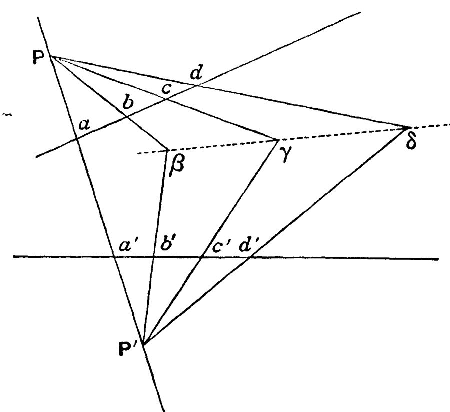

Fig. 20.

Let $(abcd\ldots)$ and $(a'b'c'd'\ldots)$ be the given homographic ranges. Then if on the line joining any pair of corresponding points a, a' we take two points P, P', and form the pencils $P(abcd\ldots)$, $P'(a'b'c'd'\ldots)$, the pairs of corresponding rays of the two pencils will intersect in a series of collinear points β, γ, $\delta\ldots$ by Art. 25, for they are homographic, and have a *common ray.*

47. If we have two homographic pencils $P(ABC\ldots)$, $P'(A'B'C'\ldots)$, and if through the point of intersection of a pair of corresponding rays we draw two transversals meeting the rays of the pencils in the two ranges $abc\ldots$, $a'b'c'\ldots$, then since the ranges are homographic and have a *common point*, the lines aa', bb', $cc'\ldots$ are concurrent by Art. 41 (2).

48. *Given a pencil of four rays, and of a second pencil three rays which correspond to three rays of the first pencil, it is required*

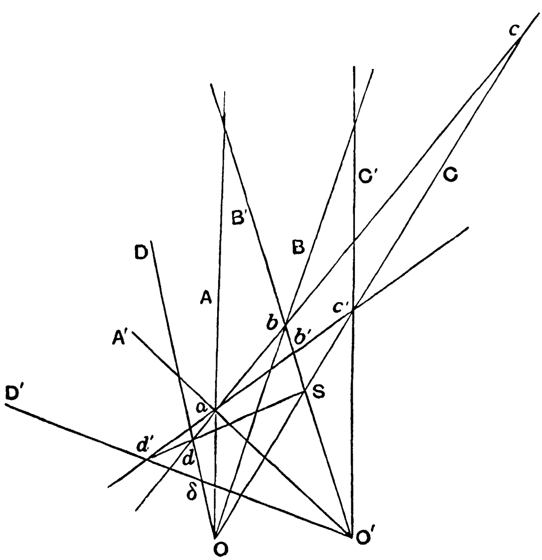

Fig. 21.

to find a fourth ray of the second pencil corresponding to the fourth ray of the first so that the two pencils may be equicross.

Let OA, OB, OC, OD be the rays of the first pencil, $O'A'$, $O'B'$, $O'C'$ the three rays of the second corresponding to OA, OB, OC. It is required to find a fourth ray $O'D'$ so that the pencils $O(ABCD)$ and $O'(A'B'C'D')$ may be equicross.

Let a be the point where OA, $O'A'$ intersect. Through a draw a transversal passing through the point b where OB, $O'B'$ intersect, and let it meet OC in c and OD in d. Through a draw a second transversal passing through the point c' where OC, $O'C'$ intersect, and let it meet $O'B'$ in b'. Then b and b' both lie on $O'B'$, and c, c' both lie on OC. Let S be the point where these two lines intersect. Join Sd, and let it cut the transversal $ab'c'$ in d'. Then $O'd'$ is the ray required.

For $(abcd)$ and $(ab'c'd')$ are the ranges in which two transversals are cut by a pencil, centre S, and are therefore equicross, by Art. 21.

49. There are two classes of questions in the solution of which the properties of homographic ranges or pencils in perspective are immediately applicable, viz.:

(1) Those in which it is required to prove that the locus of a moving point is a straight line;

(2) Those in which it is required to prove that a moving straight line passes through a fixed point.

In (1) we obtain two homographic pencils having a *common ray*, viz. the line joining the vertices, and having the different positions of the moving point for the intersections of pairs of corresponding rays.

In (2) we obtain two homographic ranges having a *common point*, and having the moving line in its different positions joining pairs of corresponding points.

EXAMPLES.

1. A line is drawn in a given direction, and is terminated by two given lines. Find the locus of the point which divides it in a given ratio.

Let the moving line in any position meet the given lines OA, OB in the points m, m', and let mm' be divided at P so that $mP = K \cdot m'P$ where K is constant. Then the different positions of the moving line constitute a pencil of parallel rays whose centre is at infinity, and therefore the ranges (m), (m') are homographic. Let $m_1P_1m_1'$ be any other position of the moving line. Then the range $(mm'P\infty) = \frac{mP}{m'P} = K =$ the range $(m_1m_1'P_1\infty)$.

The ranges being equicross, and having ∞ for a *common point*, are in perspective, and therefore PP_1, mm_1, and $m'm_1'$ are concurrent, *i.e.* the locus of P is a straight line passing through O.

2. Given two lines Oa, Ob, and two fixed points a, b on them, and also two variable points m, m' on them such that $Om + Om' = Oa + Ob$. Find the locus of the intersection of am' and bm.

By the given condition $am = bm'$, and the ranges (m), (m') being identical are homographic. Therefore the pencils $b(m)$, $a(m')$ are homographic. And when m is at a, m' is at b, and consequently ab is a *common ray* and the pencils are in perspective. Therefore the required locus is a straight line.

3. A good illustration of the method is afforded by the following proposition, which is the only one of Euclid's Porisms which has come down to us in a complete form. See Appendix I. As given by Pappus the

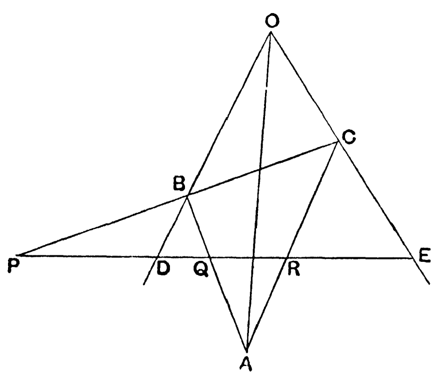

Fig. 22.

enunciation is somewhat different, but for our purpose it may be stated as follows.

Given a variable triangle ABC whose sides pass through three fixed collinear points P, Q, R. If the vertices B and C move along the given lines OD, OE, the vertex A will also describe a straight line passing through O.

By Art. 41 (1) the ranges (B) and (C) are homographic. Therefore by Art. 15 and Def. Art. 42 the pencils $Q(B)$ and $R(C)$ are homographic, and the corresponding rays QD and RE are *common rays*.

Hence by Art. 45 the pencils $Q(B)$ and $R(C)$ are in perspective. Therefore the point A lies on a fixed straight line.

Also, since O is a *common point* of the ranges (B) and (C), the locus of A passes through O.

It is easily seen that the above is equivalent to the property proved in Art. 26, viz. *Co-axial triangles are co-polar.*

4. In Ex. 3 if the points Q, R instead of being collinear with P, are collinear with O, find the locus of A.

Drawing a figure we see that the ranges (B), (C) are homographic, with O for a *common point*, and therefore the pencils $Q(B)$ and $R(C)$ are homographic with QOR for *common ray*, and the pencils are in perspective. Hence the point A lies on a straight line, which, however, does not pass through O. The student should draw the line which is the locus of A in this case.

5. Given a variable triangle ABC, two of whose sides pass through the fixed points P, Q. If the vertices move along three concurrent lines OD, OE, OF, the third side will pass through a fixed point R collinear with P and Q.

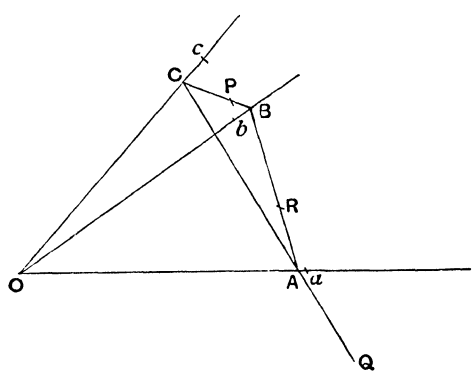

Fig. 23.

The ranges (A) and (B), by Art. 41 (1), are each homographic with the range (C), and therefore with one another, by Art. 39.

Also, when C is at O, A is at O, and B is at O, and the ranges (A) and (B), having O for *common point*, are in perspective, *i.e.* the line AB passes through a fixed point.

If PQ meets OA in a, OB in b, and OC in c, then when A is at a, C is at c, and B is at b. Therefore PQ is one of the positions of the base, and consequently the fixed point through which the base passes is collinear with P and Q.

This proposition may be stated *Co-polar triangles are co-axial.*

6. In Ex. 5 if the vertex C moves along a straight line which does not pass through O, and if the points P, Q are collinear with O, shew that AB passes through a fixed point.

Let the line OPQ meet the locus of C in S. Then when A is at O, C is at S, and B is at O, and the ranges (A) and (B) are in perspective as before.

If the locus of C meets OA in α and OB in β, the fixed point R through which AB passes is the intersection of $P\alpha$, $Q\beta$. The student should draw a figure and write out the complete proof.

For an analytical treatment of these examples, see Salmon's *Conics*, pp. 39—48.

7. Three points F, G, H are taken on the side BC of a triangle ABC; through G any line is drawn cutting AB and AC in L and M respectively. FL and HM intersect in K. Prove that K lies on a fixed straight line passing through A.

Outline of Proof. (L) and (M) are homographic ranges. $F(L)$ and $H(M)$ are homographic pencils having FH for *common ray*.

8. ABC is an isosceles triangle, and on the equal sides AB, AC equilateral triangles ABD, ACE are described. BD, CE meet in F, and BE, CD meet in G. Shew that A, F, G are in a straight line.

9. A point P, capable of moving along a given straight line, is joined to two fixed points B, C, and the lines PB, PC intersect another given straight line in X and Y. Prove that the locus of the intersection of BY and CX is a straight line.

10. A, D, C are three fixed points on a given straight line. CE is any other fixed line through C, E is a fixed point, and B is any moving point on CE. The lines AE and BD intersect in Q, the lines CQ and DE in R, and the lines BR and AC in P. Prove that P is a fixed point as B moves along CE.

CHAPTER V

CROSS-AXIS AND CROSS-CENTRE

50. *Given three points a, b, c on a line L, and also three points a', b', c' on a line L', the points α, β, γ in which the pairs of lines $(bc', b'c)$, $(ca', c'a)$, $(ab', a'b)$ intersect are collinear.*

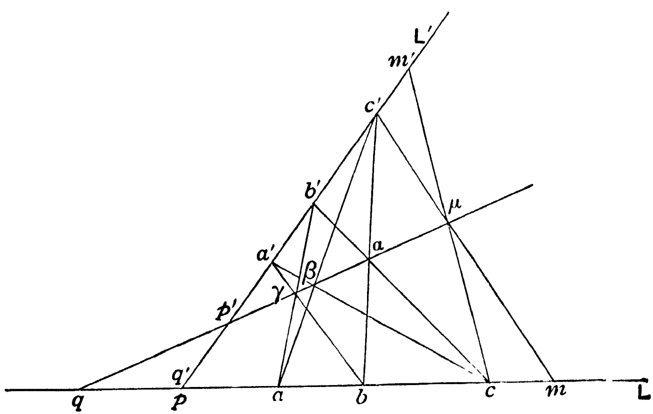

Fig. 24.

Let the lines L, L' intersect in a point which, considered as a point on L, we will denote by p, and considered as a point on L' we will denote by q'. Consider the three points a', b', c' on L' as corresponding to a, b, c on L. By Art. 40 or 46 find p' the point on L' corresponding to the point p on L, and find q the point on L corresponding to the point q' on L'.

Then $(pqab) = (p'q'a'b')$. Therefore the pencils $a'(pqab)$ and

$a\,(p'q'a'b')$ are equicross, and have the common ray aa', and therefore by Art. 25 the points p', q, γ are collinear, *i.e.* the point γ lies on the line $p'q$. By a similar reasoning we see that the points α, β lie on the same line $p'q$.

Cross-axis. DEF. The fixed line $p'q$ is called the *homographic* or *cross-axis** of the two ranges.

51. The proposition of the preceding article is one of considerable historical interest. If we join the pairs of points in the order ab', $b'c$, ca', $a'b$, bc', $c'a$, we obtain the hexagon $ab'ca'bc'$, whose vertices lie by threes on the pair of lines L, L', and whose opposite sides, taken in pairs, are $(ab', a'b)$, $(bc', b'c)$, $(ca', c'a)$, of which α, β, γ are the points of intersection. Stated in other words, the proposition tells us that if the vertices of a hexagon lie by threes on two straight lines, the points in which its opposite sides intersect lie on a straight line, being, in effect, the Pascal line of the hexagon inscribed in a line pair. This property was probably known to Euclid (300 B.C.), and employed by him without proof in his Treatise on Porisms. 600 years afterwards Pappus supplied a proof depending on Menelaus' Theorem. Thirteen centuries afterwards, in 1640, Pascal enunciated a similar theorem without proof as a property of a hexagon inscribed in a circle, and it was only after another interval of 166 years that its correlative was discovered for the conic by Brianchon in 1806.

52. *If m, m' are any pair of points on the lines L, L' such that $(abcm) = (a'b'c'm')$, and if mc', $m'c$ meet in μ, the point μ will lie on the cross-axis.*

For the pencils $a'(abcm)$ and $a\,(a'b'c'm')$ have the same cross-ratio and the common ray aa'. Therefore by Art. 25 their corresponding rays intersect on a straight line. Now the pairs of rays $(ab', a'b)$ and $(ac', a'c)$ intersect on the line $\gamma\beta$. Hence the

* Suggested by Dr Filon, *Projective Geometry*, Pref. v, 1908.

intersection of $(cm', c'm)$ lies on the same straight line, which, as we have seen, depends solely on the positions of the characteristics abc, $a'b'c'$.

• By taking (a, a') or (b, b') as centres of the pencils we see that the pairs of lines $(am', a'm)$ and $(bm', b'm)$ also intersect on the line $\alpha\beta\gamma$.

53. The existence of the cross-axis is of supreme importance, for it gives us a simple means of dividing a line L' homographically to a given divided line L. For we have merely to take any three points a', b', c' on L' to correspond to a, b, c any three given points on L, and construct the cross-axis. Then to find the point on L' corresponding to any point m on L join ma', cutting the cross-axis in μ. Then $a\mu$ will cut L' in the required point m'.

54. *By means of the cross-axis to find the point J' on L' corresponding to the point at infinity on L.*

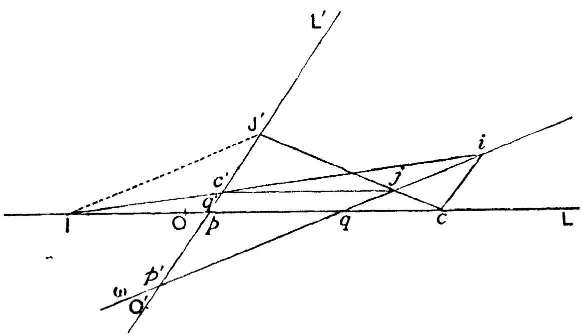

Fig. 25.

If c, c' are any pair of corresponding points, through c' draw a line parallel to L meeting the cross-axis $p'q$ in j. Then cj will meet L' in J', the point required.

To find the point I on L corresponding to the point at infinity on L', draw ci parallel to L', meeting the cross-axis in i. Then $c'i$ will meet L in the required point I.

55. *The line joining the points I, J', which correspond to the points at infinity, is parallel to the cross-axis.*

Denoting the points at infinity on L and L' by ∞ and ∞', since

$$(pqI\infty) = (p'q'\infty'J'),$$

$$\therefore \frac{pI}{qI} = \frac{q'J'}{p'J'},$$

$\therefore$ IJ' is parallel to $p'q$.

This is also obvious from Art. 52 which shews that IJ' and $\infty'\infty$ intersect on the cross-axis.

56. The points I and J' can also be found by moving one of the lines, as L', parallel to and along itself until the point p' coincides with its correspondent p. The ranges are then in

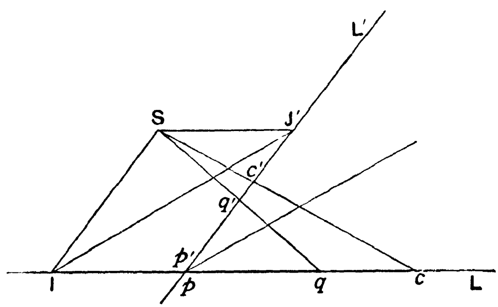

Fig. 26.

perspective, Art. 41 (2), and if S is the centre of perspective, I and J' are obtained by drawing through S parallels to L' and L. Cf. Art. 40 and Fig. 18.

The cross-axis in the Fig. 26 is the line through (p, p') parallel to IJ' by the preceding Art.

57. In the following articles we will establish a property of homographic pencils which is similar to the cross-axis property

of homographic ranges proved in Art. 50, and may be called the cross-centre property of homographic pencils, viz.:

Given three rays OA, OB, OC of one pencil, and also three rays $O'A'$, $O'B'$, $O'C'$ of a second pencil, the line joining the points of intersection of OB, $O'C'$ and OC, $O'B'$, and the line joining the points of intersection of OC, $O'A'$ and OA, $O'C'$ are concurrent with the line joining the points of intersection of OA, $O'B'$ and OB, $O'A'$.

To establish this we shall prove that these lines are each concurrent with the lines of the two pencils which correspond to $O'O$ and OO', so that the point of concurrency is a fixed point, which is called the cross-centre of the two pencils in analogy to the cross-axis of two ranges.

58. *Given two homographic pencils, vertices O, O', required to find*

(1) *The ray in the second pencil corresponding to the ray OO' in the first;*

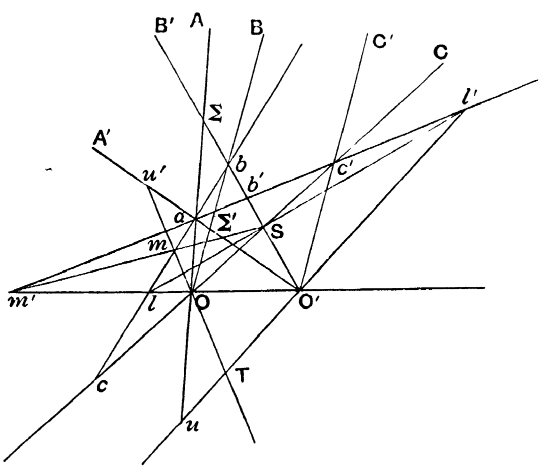

Fig. 27.

(2) *The ray in the first pencil corresponding to the ray $O'O$ in the second.*

We will employ the method of Art. 48.

Let OA, OB, OC be three rays of the first pencil, and $O'A'$, $O'B'$, $O'C'$ their corresponding rays in the second. Let OA, $O'A'$ meet in a, OB, $O'B'$ in b, OC, $O'C'$ in c', OC, $O'B'$ in S. Join ab, meeting OC in c, and OO' in l, and join ac' meeting $O'B'$ in b', and OO' in m'.

(1) Join Sl meeting ac' in l'. Then $O'l'$ is the ray corresponding to OO'. For $(abcl)$ and $(ab'c'l')$ are the ranges in which the two transversals ab, ac' are cut by a pencil, centre S, and are therefore equicross by Art. 21.

(2) If ac' meets OO' in m', and Sm' meets ab in m, then Om is the ray corresponding to $O'O$.

Let the rays Om, $O'l'$ meet in T. Then T is a known point, and has the important properties given in the following Arts.

59. *Given two homographic pencils $O(ABC\ldots)$ and $O'(A'B'C'\ldots)$, if any two non-corresponding rays OA, $O'B'$ intersect in Σ, and the rays OB, $O'A'$ intersect in Σ', then $\Sigma\Sigma'$ will pass through the fixed point T.*

For the pencils $O(ABTO')$ and $O'(A'B'OT)$ are equicross. Therefore if we cut them respectively by the transversals $O'A'$ and OA, the ranges $(a\Sigma'u'O')$ and $(a\Sigma Ou)$ are equicross, and since they have a *common point* a, they are in perspective by Art. 23. And since Ou', $O'u$ intersect in T, the line joining $\Sigma\Sigma'$ must also pass through the same point T.

60. DEF. The fixed point T may be called the *homographic* or *cross-centre**.

Cross-centre.

Since OA, $O'B'$ are any two non-corresponding rays, the position of T can be determined by means of the characteristics OA, OB, OC, and $O'A'$, $O'B'$, $O'C'$. For if OB,

* See note p. 40.

$O'C'$ intersect in S', and OC, $O'B'$ in S, the point of intersection of SS' and $\Sigma\Sigma'$ is T, the cross-centre.

61. The cross-centre enables us in theory, at any rate, to solve the problem *to construct a pencil homographic to a given pencil* in a very simple manner. Thus, to find the ray corresponding to any given ray OM, let OM meet any ray $O'A'$ in S. Join TS meeting OA in S'. Then $O'S'$ is the ray required.

The only difficulty with this method is that the cross-centre is often a distant point, or that the construction lines so frequently intersect at an inconvenient distance. Consequently the method of Art. 48 is in practice usually more convenient.

If the cross-centre method is used, the best way to determine the position of T is that given in Art. 60.

CHAPTER VI

METRICAL PROPERTIES OF HOMOGRAPHIC RANGES. THE CONSTANT OF CORRESPONDENCE. HOMOGRAPHIC EQUATIONS. ONE-TO-ONE CORRESPONDENCE

62. If m, m' are any variable pair of corresponding points,

$$(abcm) = (a'b'c'm').$$

$$\therefore \quad \frac{ac}{am} : \frac{bc}{bm} = \frac{a'c'}{a'm'} : \frac{b'c'}{b'm'},$$

$$\therefore \quad \frac{am}{bm} : \frac{a'm'}{b'm'} = \frac{ac}{bc} : \frac{a'c'}{b'c'}.$$

Let us denote this compound ratio $\frac{ac}{bc} : \frac{a'c'}{b'c'}$ between segments given by the characteristics by the letter μ.

Then
$$\frac{am}{bm} = \mu \cdot \frac{a'm'}{b'm'}.$$

Therefore the equation $\frac{am}{bm} = \mu \cdot \frac{a'm'}{b'm'}$, in which μ is a constant, represents two homographic ranges in which a, a' and b, b' are two pairs of corresponding points.

From this equation we can find any number of points m' on the second range corresponding to given points m on the first.

Here the character of the homography is given either by (1) (a, a'), (b, b') and a third pair of points (c, c'); or by (2) (a, a'), (b, b') and the value of μ; in fact, we only require to know the

corresponding lengths ab, $a'b'$ on the two ranges, and the value of μ.

Hence, when two straight lines are divided homographically, the ratio of the distances of any point of division m from two fixed points on the first is equal to the ratio of the distances of the corresponding point of division m' from the two corresponding fixed points on the second range multiplied by a constant. Conversely,

When two variable points m, m' divide two fixed segments ab, $a'b'$ on two lines L, L' in such a manner that $\frac{am}{bm} = \mu \cdot \frac{a'm'}{b'm'}$, where μ is a constant, L and L' are divided homographically by the points m, m'.

63. Given abc, $a'b'c'$ the characteristics of two homographic ranges, the positions of I, J' can be obtained metrically by actual calculation of the lengths aI, $a'J'$ as follows:

$$(abcI) = (a'b'c'\infty'),$$

$$\therefore \frac{ac}{aI} : \frac{bc}{bI} = \frac{a'c'}{a'\infty'} : \frac{b'c'}{b'\infty'} = \frac{a'c'}{b'c'},$$

$$\therefore \frac{aI}{bI} = \frac{ac}{bc} : \frac{a'c'}{b'c'} = \mu,$$

$$\therefore aI = \mu bI = \mu(aI - ab),$$

$$\therefore aI(\mu - 1) - \mu ab,$$

$$\therefore aI = \frac{\mu}{\mu - 1} \cdot ab.$$

In the same way it will be found that

$$\frac{a'J'}{b'J'} = \frac{1}{\mu}, \qquad \therefore a'J' = \frac{a'b'}{1 - \mu}.$$

Hence $\quad bI = \frac{aI}{\mu} = \frac{ab}{\mu - 1}$, and $b'J' = \mu a'J' = \frac{\mu}{1 - u} \cdot a'b'$.

64. *If m, m' are a pair of corresponding points of the ranges, to prove that $Im \,.\, J'm'$ is constant.*

Since $$(aIm\infty) = (a'\infty' m'J'),$$

$$\therefore \frac{am}{Im} = \frac{a'm'}{a'J'},$$

$$\therefore (Im - Ia)\, J'a' + (J'm' - J'a')\, Im = 0,$$

$$\therefore Im \,.\, J'm' = Ia \,.\, J'a' = \text{const. } \lambda \text{ (say)}.$$

65. The converse of the property found in the preceding article is a very important one, and tells us that

If on two given straight lines we take fixed points I, J', and also two variable points m, m' such that $Im \,.\, J'm'$ is constant, then the ranges (m) and (m') will be homographic, and the points I, J' will correspond to the points at infinity in the two ranges.

The constant λ is called by Steiner "the power of the correspondence," and its actual value is seen from Art. 63 to be $-\frac{\mu}{(1-\mu)^2} \,.\, ab \,.\, a'b'$, where $\mu = \frac{ac}{bc} : \frac{a'c'}{b'c'}$.

It should be noticed that λ is an absolute constant, which holds for all pairs of corresponding points on the ranges, whilst μ depends on the characteristics abc, $a'b'c'$, so that if we take two fixed points on the lines and call them I and J', all that we require to fix the homography is the value of λ, whereas, if we employ μ, we must know the values of ab, $a'b'$ and μ.

Homographic Equations.

66. The relation between any pair of corresponding points m, m' in two homographic ranges can, as we have shewn, be expressed by equations, of which the simplest form is

I. $Im \,.\, J'm' = Ia \,.\, J'a'$ ....(1) Art. 64.

Here the origins are I, J'.

If the ranges are on two separate lines, the points I, J' may be at their point of intersection. If the ranges are co-axial, and I, J' coincide, we have the case of involution, Chap. IX.

II. The next in order of simplicity is

$$\frac{am}{bm} = \mu \cdot \frac{a'm'}{b'm'} \quad \text{..............(2) Art. 62.}$$

Here there are two origins on each line, viz. a, b; a', b'.

III. From (1) we can obtain other forms which we shall find useful, in which there is a single origin taken on each line, consisting of any chosen pair of non-corresponding points which we shall call a, b'. The variable quantities in these equations will be am, $b'm'$, where m, m' are a variable pair of corresponding points. If a, a' and b, b' are pairs of corresponding points, writing $Im = am - aI$, and $J'm' = b'm' - b'J'$, we have

$$(am - aI)(b'm' - b'J') = aI \,.\, a'J',$$

$$\therefore \quad am \,.\, b'm' - b'J' \,.\, am - aI \,.\, b'm' + aI \,.\, b'J' = aI \,.\, a'J',$$

and

$$b'J' - a'J' = b'a',$$

$$\therefore \quad am \,.\, b'm' - b'J' \,.\, am - aI \,.\, b'm' + aI \,.\, b'a' = 0 \text{......(A).}$$

Again, since by Art. 64

$$\frac{aI}{b'J'} = \frac{bI}{a'J'} = \frac{aI - bI}{b'J' - a'J'} = \frac{ab}{b'a'},$$

therefore

$$aI \,.\, b'a' = b'J' \,.\, ab.$$

Hence the relation (A) may be written

$$am \,.\, b'm' - b'J' \,.\, am - aI \,.\, b'm' + b'J' \,.\, ab = 0 \text{ ......(B).}$$

From Art. 63, putting $aI = \dfrac{\mu}{\mu - 1} \,.\, ab$, $b'J' = \dfrac{\mu}{\mu - 1} \,.\, b'a'$ in (A) and (B), these become

$$(1 - \mu) am \,.\, b'm' + \mu \,.\, b'a' \,.\, am + \mu \,.\, ab \,.\, b'm' - \mu \,.\, ab \,.\, b'a' = 0 \text{...(3),}$$

where μ is an absolute number which depends solely on the characteristics of the ranges, and can have any values except zero and infinity.

IV. If the origins coincide at the intersection of the ranges (when it is not a *common point*), *i.e.* when b' coincides with a, we have merely to write a for b' in (3) and we obtain

$$(1-\mu)\, am\,.\,am' + \mu aa'\,.\,am + \mu ab\,.\,am' - \mu ab\,.\,aa' = 0 \ldots (4),$$

or
$$am\,.\,am' - aJ'\,.\,am - aI\,.\,am' + aI\,.\,aa' = 0 \quad \ldots\ldots (4').$$

The ranges are not in perspective.

NOTE. In equations (4) and (4′) the quantities am', aa', aJ' could not exist unless a were on the line L', as all measurements are between points on the same line. Hence in these quantities a stands for the coincident point b'.

In all the above equations, except (2) where four origins are used, the origins have been non-corresponding points.

V. If the origins are required to be a pair of corresponding points a, a', we shall find it best to go back to (1), and write in it $Im = am - aI$, $J'm' = a'm' - a'J'$, from which we shall obtain

$$am\,.\,a'm' - a'J'\,.\,am - aI\,.\,a'm' = 0 \ \ldots\ldots\ldots\ldots (5'),$$

or
$$(1-\mu)\, am\,.\,a'm' - a'b'\,.\,am + \mu ab\,.\,a'm' = 0 \ \ldots\ldots\ldots (5).$$

The ranges may be, but are not necessarily, in perspective.

VI. If the ranges are in perspective, so that their intersection is a *common point*, and if this point is taken as the common origin a, a', then writing a for a' in (5) we have

$$(1-\mu)\, am\,.\,am' - ab'\,.\,am + \mu ab\,.\,am' = 0 \ \ldots\ldots\ldots (6),$$

or
$$am\,.\,am' - aJ'\,.\,am - aI\,.\,am' = 0 \ \ldots\ldots\ldots\ldots (6'),$$

and the line mm' passes through a fixed point.

It is interesting to notice that the coordinates of this fixed point, referred to the two lines as axes, are $(aI,\ aJ')$. For if $(X,\ Y)$ is on mm', we have

$$\frac{X}{am} + \frac{Y}{am'} = 1,$$

i.e.
$$am\,.\,am' - Y\,.\,am - X\,.\,am' = 0,$$

whence $X = aI$, and $Y = aJ'$.

This is, in fact, only a restatement of what we have already shewn geometrically in Art. 56.

From V and VI we see that

• *If the origins are a pair of corresponding points, the homographic equation has no absolute term, and, conversely, if the homographic equation has no absolute term, the origins are a pair of corresponding points.*

Also, when the intersection of the ranges is taken as the common origin, if the homographic equation has no absolute term, the ranges are in perspective.

It only remains for us to consider the case when $\mu = 1$.

Proportional Section*.

67. When $\mu = 1$, the equation (2) of Art. 66 becomes

$$\frac{am}{a'm'} = \frac{bm}{b'm'} = \frac{ab}{a'b'} = \text{const.}$$

Therefore the lines are divided similarly, or into proportional parts. And since, by Art. 63, $\frac{aI}{bI} = \mu = 1$, the point I must be a point at infinity on the line L. Similarly J' is a point at infinity on the line L'. So, conversely

If two straight lines are divided into proportional parts, they are divided homographically, and if the points I, J' are at infinity, the two ranges are similar.

Of course, since I is by definition the correspondent of a point at infinity, the condition that I should be at infinity is equivalent to the condition that the ranges should have a pair of corresponding points at infinity, and so for J'.

* On this form of homographic division Apollonius wrote his treatise *de Sectione rationis* in two books containing 181 propositions. This work, which was extant in Greek at the time of Pappus, was discovered in an Arabic MS and translated into Latin by Halley in 1706. See Art. 88.

68. If we put $\mu = 1$ in the other equations in Art. 66,

VII. (3) becomes $\dfrac{am}{ab} + \dfrac{b'm'}{b'a'} = 1$ ....(7),

the origins being the non-corresponding points a, b'.

VIII. (4) becomes $\dfrac{am}{ab} + \dfrac{am'}{aa'} = 1$ ....(8),

the origins being at the intersection of the ranges, which, however, is not a *common point*, and the ranges are not in perspective.

IX. (5) becomes $\dfrac{am}{ab} = \dfrac{a'm'}{a'b'}$ ....(9),

the origins being the corresponding points a, a'.

The ranges may be, but are not necessarily, in perspective.

X. (6) becomes $\dfrac{am}{ab} = \dfrac{am'}{ab'}$................(10),

the common origin being the intersection of the ranges, which is a *common point*, and the ranges are in perspective.

If the lengths of the segments from m and m' to the origins are denoted by x, x', the equations in Arts. 66 and 68 are of the form

$$Axx' + Bx + Cx' + D = 0,$$

and may be divided into two classes. In the one in which the term xx' occurs the homography may be said to be of the second order, and in the other, where the term xx' is wanting, it may be said to be of the first order, the ranges being then divided similarly or proportionally.

In homographic equations of the second order,

(1) *If the origins are non-corresponding points*
neither A nor D can $= 0$.

If $C = 0$, the origin for x is at I.

If $B = 0$, the origin for x' is at J'.

If $B = 0$ and $C = 0$, the origins are at I, J'.

(2) *If the origins are corresponding points*
$D = 0$, but neither A, B, nor C can vanish, and the equation must be of the form $Axx' + Bx + Cx' = 0$.

In homographic equations of the first order,

(3) *If the origins are non-corresponding points*

neither B, C, nor D can vanish (for then an origin would be at infinity), and the equation is of the form $Bx + Cx' + D = 0$.

(4) *If the origins are corresponding points*

$D = 0$, and the equation is of the form $Bx + Cx' = 0$, and we draw the same conclusions as in the case of homographic equations of the second order given at the end of Art. 66.

One-to-One Correspondence.

69. In both orders of equations corresponding to any given value of x there is one and only one value of x', and corresponding to a given value of x' there is one and only one value of x. When this is the case x and x' are said to be connected by a one-to-one or (1, 1) correspondence, and the ranges marked out on the two lines by giving different values to x or x' are homographic.

It should be noticed, however, that for the equation

$$Axx' + Bx + Cx' + D = 0$$

to give two homographic ranges, we must not have $A : B = C : D$.

For in that case the expression on the left hand could be factorised, and we should have $(Ax + C)\left(x' + \dfrac{B}{A}\right) = 0$, and therefore one of the two variables must have a certain definite value, the other being then free to take *any* value, *i.e.* in a factorising homography, all points of either line correspond to a single point of the other line*.

The following examples are intended to illustrate the meaning and application of the homographic equations given in Arts. 66—68. If the relation between a variable pair of points on two straight lines is of either of the types (6) or (10), their intersec-

* For a geometrical illustration of this, see an article on "The double six" by G. T. Bennett, in *Proceedings of the London Mathematical Society*, p. 336, April 25th, 1911.

tion being a common origin, then when $x = 0$, x' also $= 0$ and the intersection of the ranges is a *common point.* The ranges are then in perspective, and the lines joining pairs of corresponding points pass through a fixed point.

If the equation expressing the relation is of any of the types (3) to (10), and from the ranges pencils can be formed having the line joining the vertices for a *common ray*, the pencils are in perspective, and the locus of the intersections of pairs of corresponding rays is a straight line.

It will be seen in Chap. XI that if the ranges are homographic but not in perspective the lines joining pairs of corresponding points will envelop a conic touching the ranges; and if the pencils are not in perspective the locus of the intersections of pairs of corresponding rays is a conic passing through the centres of the pencils.

In forming the homographic equation connecting two ranges we can generally determine its order by inspection from the consideration that if the points I, J' are at a finite distance the equation is of the second order, whilst it is of the first order if they are at infinity.

EXAMPLES.

1. A line through a fixed point P on the base BC of a triangle ABC cuts the sides AB, AC in points m, m'. Find the homographic equation for the ranges (m) and (m') taking (1) A as common origin, (2) B and C as origins, and deduce from them the positions of the points I, J' on the sides AB, AC.

(1) Let $Am = x$, $Am' = x'$. Then by Menelaus' Theorem, Art. 18,

$$Am' \,.\, CP \,.\, Bm = Cm' \,.\, BP \,.\, Am \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

$$\therefore\ x' \,.\, CP \,.\, (x - c) = (x' - b) \,.\, BP \,.\, x,$$

$$\therefore\ xx'\,(BP - CP) - b \,.\, BP \,.\, x + c \,.\, CP \,.\, x' = 0,$$

$$\therefore\ xx' - \frac{b}{a} \,.\, BP \,.\, x + \frac{c}{a} \,.\, CP \,.\, x' = 0.$$

Comparing this with the homographic equation (6′) of Art. 66, VI, we deduce that $AJ' = \frac{b}{a} \,.\, BP$; $AI = -\frac{c}{a} \,.\, CP$, as is of course obvious from the geometry of the figure.

(2) Let $Bm=x$, $Cm'=x'$. Substituting in (A) we have

$$(b+x')\,CP\,.\,x=x'\,.\,BP\,(x+c),$$

$$\therefore\ xx'\,(BP-CP)-b\,.\,CP\,.\,x+c\,.\,BP\,.\,x'=0,$$

$$\therefore\ xx'-\frac{b}{a}\,.\,CP\,.\,x+\frac{c}{a}\,.\,BP\,.\,x'=0,$$

$\therefore$ by Art. 66, V, (5′) $\quad CJ'=\frac{b}{a}\,.\,CP\,;\ BI=-\frac{c}{a}\,.\,BP.$

We might also have approached the question in the following manner.

(1) The ranges (m) and (m') are obviously two homographic ranges of the second order and in perspective. Therefore by Art. 66, VI their homographic equation is

$$xx'-AJ'\,.\,x-AI\,.\,x'=0,$$

and from the geometry of the figure

$$AJ'=\frac{b}{a}\,.\,BP,\quad AI=-\frac{c}{a}\,.\,CP.$$

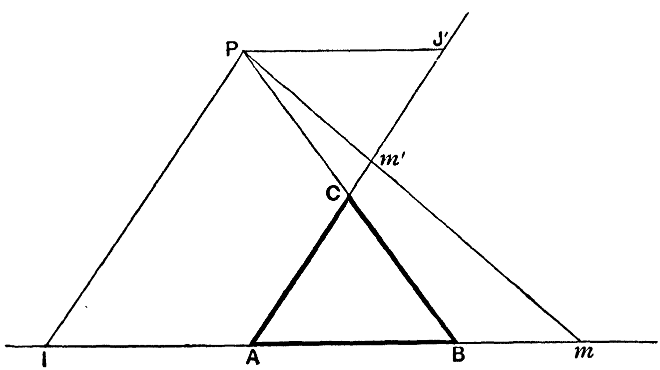

Fig. 28.

(2) When B and C are origins, by Art. 66, V, the homographic equation is

$$xx'-CJ'\,.\,x-BI\,.\,x'=0,$$

and $\quad CJ'=\frac{b}{a}\,.\,CP,\quad BI=-\frac{c}{a}\,.\,BP.$

2. Through the angle C of a parallelogram $ABCD$ a straight line is drawn meeting the two sides AB, AD in a, a'. Prove that the rect. $Ba\,.\,Da'$ is constant.

The ranges (a), (a') are obviously homographic. When a' is at infinity, a is at B, and when a is at infinity, a' is at D. Therefore by Art. 64 $Ba \,.\, Da'$ is constant.

3. AB, AC are two given straight lines of lengths a, b, in which points P and Q are taken such that $AP : AB = AQ : QC$. Prove that the straight line PQ passes through a fixed point.

$$AP : AB = AQ : QC$$
$$= AQ : AC - AQ,$$
$$\therefore\ AP\,(AC - AQ) = AB \,.\, AQ,$$
$$\therefore\ x\,(b - x') = ax', \quad \text{where } AP = x, \quad AQ = x',$$
$$\therefore\ xx' - bx + ax' = 0,$$

$\therefore$ by Art. 66 (6) (P) and (Q) are homographic ranges, the origin A being a *common point*. They are therefore in perspective, and PQ passes through a fixed point.

4. In Ex. 3 if D is the mid-point of AC, shew that the fixed point is the intersection of BD with the line through C parallel to AB. Prove that if E is taken on AB such that $AE = -AB$, the fixed point lies on the parallels to the given lines drawn through E and C. Shew also that E coincides with I and C with J'.

5. Given the base AB of a triangle ABC, and the length of the segment mm' which the sides intercept on a straight line PQ parallel to AB, shew that the locus of the vertex C is a straight line.

(m) and (m') are homographic ranges, being identical, and $A\,(m)$, $B\,(m')$ are homographic pencils. When m is at infinity, so also is m'. Hence AB is a *common ray*, the pencils are in perspective, and the locus of C is a straight line.

What is the force of the limitation that PQ is parallel to AB?

6. In Ex. 5 if a and b are fixed points on PQ, and the ratio of the segments am, bm' is given, shew that the locus of the vertex is a straight line.

Let $am = k \,.\, bm'$. Then by Art. 67, (m) and (m') are homographic ranges, &c.

7. Q, R are fixed points in BC, the base of a triangle ABC. A line mm' parallel to the base meets the sides AB, AC in m, m'. Shew that the locus of the intersection of Qm, Rm' is a straight line.

$$\frac{Am}{Am'} = \frac{AB}{AC} = \text{const.},$$

$\therefore$ by Art. 68 (10) the ranges (m) and (m') are homographic, &c.

8. In Ex. 7 if Cm, Bm' meet in P, shew that the locus of P is a straight line. Shew that the same results hold in Exs. 7, 8 if mm', instead of being parallel to the base, cuts it in a fixed point.

• 9. If on AB, the base of a triangle ABC, we take any length AT, and at the other end of the base another length BS in a fixed ratio to AT, and draw ET and FS parallel to a fixed line CR, meeting CA in E and CB in F, shew that the locus of O, the intersection of EB and FA, is a straight line. (For analytical solution see Salmon's *Conics*, p. 45, Ex. 5.)

By Art. 66 the pair of ranges (T) and (S) are homographic, as are also (T) and (E) and also (S) and (F). Therefore by Art. 39 the ranges (E) and (F) are homographic, and therefore also the pencils $B\,(E)$ and $A\,(F)$.

Now when E is at A, T is at A, S is at B, and F is at B. Hence AB is a *common ray* of the pencils which are thus in perspective and the locus of O is a straight line.

10. OA, OB are two given lines, m and m' a pair of corresponding points of two ranges on them whose homographic equation is of the first order. If the perpendiculars at m and m' meet in P, shew that the locus of P is a straight line.

By Art. 68 (8) the ranges (m) and (m') are similar. The series of perpendiculars Pm constitute a pencil of parallel rays whose centre is at the point ∞, and the perpendiculars Pm' form a pencil whose centre is at ∞'. The ranges (m) and (m') being homographic, so also are the pencils $\infty\ (P)$ and $\infty'\,(P)$. Also, when m is at infinity, m' is at infinity, and the rays ∞m, $\infty' m'$ coincide in the line at infinity. Therefore the pencils, having a *common ray*, are in perspective, and the locus of P is a straight line.

11. Find the locus of the orthocentre of the triangle two of whose sides are given in position, and whose base passes through a fixed point.

Let O be the vertex of the triangle, P the fixed point, Pmm' any position of the base, H the orthocentre of the triangle Omm'. Then the ranges (m), (m') are homographic, their equation being of the second order. The series of perpendiculars mH, $m'H$ form two homographic pencils, centres ∞, ∞'. The line joining their centres is the line at infinity, but this does not pass through m and m' at the same time, for m and m' are not at infinity together, as the ranges are not similar. The pencils are therefore not in perspective.

It will be seen in Chap. XI that the locus of H is a hyperbola.

12. Two sides of a triangle are given in position, and their sum is constant. Prove that the centre of the nine-points circle traces out a straight line.

Let AOB be any position of the triangle, and let $OA=x$, $OB=x'$, so that $x+x'=$const. Let D be the mid-point of OA, BD' perpendicular to OA, and

m the mid-point of DD'. Forming m' in the same way, the perpendicular to OA through m meets the perpendicular to OB through m' in the nine-points centre.

Then
$$2Om = \frac{x}{2} + x' \cos O,$$
$$2Om' = x \cos O + \frac{x'}{2},$$
$$\therefore \quad Om + Om' = \tfrac{1}{2}(\tfrac{1}{2} + \cos O)(x + x') = \text{const.},$$

$\therefore$ by Ex. 10 the locus of the nine-points centre is a straight line.

13. Find the locus of the centre of the circum-circle of a triangle when the position and the sum or difference of two of the sides are given.

14. Given the position and the sum or difference of the reciprocals of two sides of a triangle, shew that the base will always pass through a fixed point.

15. In Ex. 14, if Om, Om' are the sides, shew that the base will pass through a fixed point if the sides are connected by the relation
$$\frac{k}{Om} + \frac{l}{Om'} = n,$$
where k, l, n are constants.

16. OA, OB are two given straight lines, A and B fixed points. The points P on OA and Q on OB vary in such a manner that
$$\frac{1}{OA} - \frac{1}{OP} = \frac{1}{OB} - \frac{1}{OQ},$$
shew that PQ passes through a fixed point.

17. OA and OB are two given straight lines, and from a fixed point C two straight lines CM, CN are drawn to them so that the triangles OMN, CMN are equal. Shew that MN passes through a fixed point.

18. OP, OQ are fixed lines, and the circum-centre of the triangle OPQ lies on another fixed line. Shew that P and Q are corresponding points of two ranges of the first order not in perspective.

19. Through a fixed point O two straight lines OPQ and $OP'Q'$ are drawn meeting two fixed parallel straight lines. If PQ' and $P'Q$ meet in R, prove that the locus of R is a straight line.

20. In Ex. 19 if the two fixed lines are not parallel, shew that the locus of R is a straight line.

21. Through a fixed point a straight line is drawn meeting two fixed parallel straight lines in P and Q respectively, and through P and Q straight lines are drawn in given directions intersecting in R. Prove that the locus of R is a straight line.

22. ABC is a triangle, MN is any straight line parallel to AC, cutting the sides BC, BA of the triangle in M and N respectively. Shew that the locus of the intersection of AM and CN is a straight line.

23. A and B are fixed points in a line, and C, D are fixed points in another line parallel to AB. Find the locus of a point P such that if PA, PB meet CD in Q and R, the sum of CQ and DR is constant.

24. L, L' are two fixed lines, ABC a triangle whose base BC, of constant length, slides along L, and the vertex A moves along L' in such a way that AB is always parallel to a given direction. If the side AC is divided in a constant ratio at K, the locus of K is a straight line. Use Chap. IV, Ex. 1.

25. L, L' are two fixed lines, m, m' are a pair of corresponding points on them in the two ranges given by the equation $Bx + Cx' + D = 0$. If mm' is joined and divided in a constant ratio at k, the locus of k is a straight line. Newton, *Princip.* Bk I, Lemma xxiii. For the case where the equation between m, m' is of the second order, see Chap. XI, Ex. 13.

26. Given in magnitude and position the vertical angle of a triangle, and the sum or difference of the sides containing it, the locus of the mid-point of the base is a straight line.

27. A parallelogram is inscribed in a triangle, having one side on the base of the triangle, and the two sides adjacent to it parallel to a fixed direction. Prove that the locus of the intersection of the diagonals of the parallelogram is a straight line bisecting the base of the triangle.

28. OA, OB are two given straight lines. The points P on OA and Q on OB vary in such a manner that the ratio of AP to BQ is constant. Shew that the locus of the mid-point of PQ is a straight line.

29. OA, OB are two given lines, m, m' a pair of points on them such that the perpendiculars to OA at m and to OB at m' meet on a fixed straight line. If through m, m' are drawn parallels to the given lines, shew that the locus of their intersection is a straight line.

30. OA, OB are two fixed lines, m, m' a pair of corresponding points on them in the ranges given by the relation $\frac{k}{Om} + \frac{l}{Om'} = n$, where k, l, n are constants. If through m, m' are drawn parallels to the given lines, discuss the question whether the locus of their intersection is, or is not, a straight line.

CHAPTER VII

TWO HOMOGRAPHIC CO-AXIAL RANGES. THEIR COMMON POINTS, AND METHODS OF FINDING THEM

70. HITHERTO we have supposed the two homographic ranges to be on two separate lines L and L', on each of which are given the three arbitrary points a, b, c and a', b', c'. Now if m is any point of the range on L, the corresponding point m' on L' is given by the relation $(m'a'b'c') = (mabc)$, which shews that the position of m' on L' is quite independent of the angle between L and L'. Hence in Fig. 24, p. 39, if we suppose L' to rotate about q' until it coincides with L, each of the points of the range on L' will describe a circle with q' as centre, and will still be at the same distance from p (or q') as before, the homography of the ranges will be unaltered, and p being now the common origin, the relation (4′) of Art. 66 representing two homographic ranges on the same straight line becomes

$$pm\,.\,pm' - pJ'\,.\,pm - pI\,.\,pm' + pI\,.\,pp' = 0.$$

Common Points.

Common Points.

71. When the ranges are on different lines their point of intersection might or might not be a *common* point of the ranges, a *common* point being defined as a point of coincidence of two corresponding points of the ranges, and of course there could not be more than the one *common* point.

When, however, the ranges are co-axial there will usually be two *common* points, but never more than two, unless the ranges coincide. These *common* points are of great interest and importance as will be shewn by examples in the following chapter.

They are obtained by putting m for m' in the relation of Art. 70, which gives us

$$pm^2 - (pI + pJ')\,pm + pI\,.\,pp' = 0.$$

This being a quadratic equation gives us two and only two values of pm, and these of course may be real and unequal, coincident, or imaginary.

If e and f are the points given by these values of pm, they are the *common points* of the ranges. Some of their properties are given in the following articles.

72. *The mid-point of ef is also the mid-point of IJ'.*

For if O is the mid-point of ef, by the equation in Art. 71 we have $2pO = pI + pJ'$.

The coincidence of the mid-points may also be proved by using Art. 64.

For since $\qquad \lambda = eI\,.\,eJ' = If\,.\,J'f,$

$$\therefore\ \frac{eI}{J'f} = \frac{If}{eJ'} = \frac{eI + If}{eJ' + J'f} = \frac{ef}{ef} = 1,$$

$$\therefore\ eI = J'f, \text{ and } If = eJ'.$$

Therefore the mid-point of ef is also the mid-point of IJ'.

73. *If O, the mid-point of ef, is a point on the first range, and O' its correspondent, the range $(efO'J')$ is harmonic.*

Transferring the origin from p to O, in which case pO is zero, we have

$$Oe^2 + OI\,.\,OO' = 0,$$

and since $OI = -OJ'$, and $Oe^2 = Of^2$, this becomes

$$Oe^2 = Of^2 = OJ'\,.\,OO' \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{A}),$$

and therefore by Art. 32 $(efO'J')$ is a harmonic range. It also follows from (A) that

The common points are real or imaginary according as OJ' and OO' have the same or opposite signs, i.e. according as J' and O' are on the same or opposite sides of O.

The above relation (A) can also be obtained by using the constant of correspondence, Art. 64.

Thus, from $\quad Im \,.\, J'm' = \text{const.} = Ia \,.\, J'a'$,

we have $\quad (Om - OI)(Om' + OI) = (Oa - OI)(Oa' + OI)$,

and therefore $Om \,.\, Om' - mm' \,.\, OI = Oa \,.\, Oa' - aa' \,.\, OI$,

as the general relation connecting the distances from O of corresponding points. In particular, for common points this becomes

$$Oe^2 = Oa \,.\, Oa' - aa' \,.\, OI,$$

shewing at once that there are two common points equidistant from O. If a coincides with O,

$$Om \,.\, Om' - mm' \,.\, OI = -OO' \,.\, OI = OO' \,.\, OJ',$$

$$\therefore\ Oe^2 = Of^2 = -OO' \,.\, OI = OO' \,.\, OJ'.$$

The cross-ratio formed by the common points and any pair of corresponding points is constant.

74. Taking the relation given in Arts. 62, 66

$$\frac{am}{bm} = \mu \frac{a'm'}{b'm'};$$

since a, a' and b, b' may be any two pairs of corresponding points, let a be the common point e, and b the common point f. Then a' coincides with e, and b' with f, and the relation becomes

$$\frac{em}{fm} = \mu \frac{em'}{fm'},$$

where μ is const.

$$\therefore\ \mu = \frac{em}{fm} : \frac{em'}{fm'} = (efmm').$$

75. The above result may also be obtained as follows.

Since $(abef) = (a'b'ef)$,

$$\therefore \frac{ae}{af} : \frac{be}{bf} = \frac{a'e}{a'f} : \frac{b'e}{b'f},$$

$$\therefore \frac{ae}{af} : \frac{a'e}{a'f} = \frac{be}{bf} : \frac{b'e}{b'f},$$

$$\therefore (aa'ef) = (bb'ef).$$

Hence the cross-ratio of the range formed by the common points and any pair of corresponding points is constant.

76. *There cannot be more than two common points.*

For suppose there were three, viz. e, f, g. Then

$$(efgm) = (efgm'),$$

therefore m' would always coincide with m, and the two ranges would be identical. See Art. 71.

77. If one of the common points, as f, is at infinity, since

$$(mea\infty) = (m'ea'\infty),$$

$$\therefore \frac{ma}{ea} = \frac{m'a'}{ea'},$$

$$\therefore \frac{ae}{a'e} = \frac{am}{a'm'} = \frac{ab}{a'b'}.$$

Therefore the lines are divided proportionally.

Conversely, if two ranges are similar, one of the common points is at infinity. See also Art. 67.

If in addition the ranges are equal and in the same sense, so that $ab = a'b'$, then $ae = a'e$, and therefore e is at infinity, *i.e.* if the ranges are superposable, both common points are at infinity.

78. By Art. 73 $Oe^2 = OJ' \,.\, OO'$.

If the point O' coincides with O, we see that Oe vanishes, the common points coincide at O, and the constant of correspondence $= Im \,.\, J'm' = IO \,.\, J'O = -\frac{1}{4} IJ'^2$.

The case where O coincides with J' belongs to the system in involution, and will be considered in Chapter IX. OO' is then infinite, whilst the product $OJ' . OO'$ is still finite.

By Art. 64
$$\begin{aligned}\lambda &= Ie \,.\, Je\\ &= (Oe - OI)(Oe + OI)\\ &= Oe^2 - OI^2,\end{aligned}$$
$$\therefore\ Oe^2 = OI^2 + \lambda = Of'^2.$$

From this we see that if λ is positive, Oe is always real.

If λ is negative, and $< OI^2$, Oe is always real.

If λ is negative and $= - OI^2$, Oe vanishes, and the common points coincide at O.

If λ is negative and $> OI^2$, the common points are imaginary.

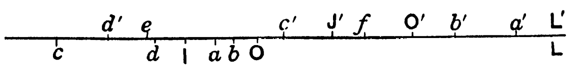

Fig. 29.

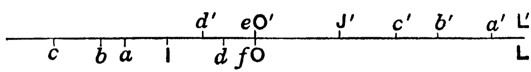

Fig. 30.

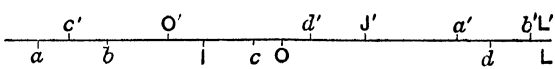

Fig. 31.

The different cases of this article are illustrated by the above figures, each of which gives two co-axial homographic ranges. The divisions of L are shewn below the line, and those of L' above it. In Fig. 29 the common points are real and separate; in Fig. 30 they are real and coincident, in which case all the points e, f, O, O' coincide. In Fig. 31 they are imaginary, although their mid-point O is real.

79. If we have two homographic ranges on the same axis, and if a is a point on the axis which has a' or a'' for its correspondent according as we consider it to be a point in the first or second range, then, as the point a varies, the range which it describes will be homographic with the range (a''); and it is by supposition homographic with the range (a'). Therefore by Art. 39, the ranges (a') and (a'') are homographic. Also, if a coincides with either of the common points of the ranges (a) and (a'), the points a' and a'' will also coincide with it. Hence the ranges (a') and (a'') will have the same common points as the ranges (a) and (a').

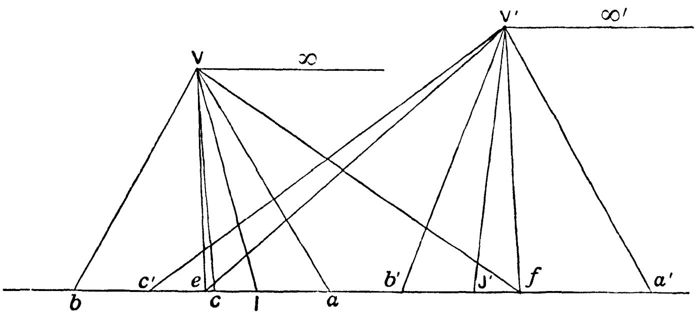

Fig. 32.

80. If two homographic pencils, vertices V, V', are cut by a transversal, we shall obtain two homographic ranges upon it, and if e, f are their common points, Ve, $V'e$ and Vf, $V'f$ are in general the only pairs of corresponding rays which intersect on the transversal. If there are more than two such pairs, the pencils are in perspective, having the transversal for axis of perspective. See Art. 45.

If we have two co-axial homographic ranges, and we join the points of division to an external point V', we shall obtain two homographic concentric pencils in which the rays drawn to the common points will be *common rays*, and the pencils will

have properties similar to those possessed by two co-axial ranges, of which the two most important are

(1) *by Art.* 74 *the cross-ratio formed by the two common rays and any pair of corresponding rays is constant,*

(2) *by Art.* 76 *there cannot be more than two common rays.*

81. *Given two homographic ranges on the same straight line, if the common points are imaginary there are two points on opposite sides of the line at which pairs of corresponding points subtend equal angles.*

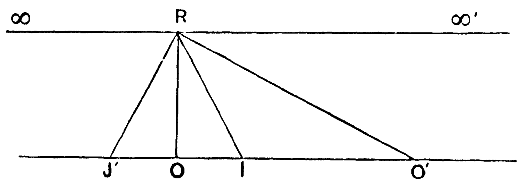

Fig. 33.

Since, by Art. 73, O' and J' are on opposite sides of O, therefore O' and I are on the same side of O, and the product $OI \,.\, OO'$ is positive. Draw OR perpendicular to the given line of such length that $OR^2 = OI \,.\, OO'$, and through R draw a line parallel to the given line.

Then $J'RO'$ is a right angle, and the right-angled triangles ORI, $OO'R$ are similar. Therefore the angle

$$ORI = OO'R = O'R\infty',$$

the angle $IR\infty = \infty'RJ'$, and the angle $OR\infty = O'RJ'$.

Therefore in the two pencils $R\,(OI\infty)$, $R\,(O'\infty'J')$ the angle between each pair of rays of the one is equal to the angle between the corresponding rays of the other taken in the same sense. Consequently by Art. 42, if m, m' are any pair of corresponding points in the given ranges, the pencils $R\,(OI\infty m)$, $R\,(O'\infty'J'm')$ are superposable by rotation through the angle ORO'.

The point R', the image of R on the other side of $O'J'$, will of course have similar properties.

Hence, if either of these points is found, and also any pair of corresponding points (m, m'), any additional number of pairs can be found by means of the constancy of the angle mRm'.

Rotation centres, Rotation angle.

DEF. We might call these points R, R' the *rotation centres* of the homography, and the constant angle between corresponding rays the *rotation angle*.

These points can be obtained without finding O and O' if we know the power of the correspondence λ, *i.e.* if we know I, J' and any pair of corresponding points (m, m'). For $OR^2 = OI \,.\, OO'$,

$$\therefore\ RI^2 = OR^2 + OI^2 = OI\,(OO' + OI) = OI\,(OO' - OJ') = OI \,.\, J'O'.$$

$$\therefore\ RI^2 = RJ'^2 = -\lambda.$$

Therefore RI and RJ' are equal to the mean proportional between mI and $J'm'$.

Another method of obtaining the positions of the rotation centres is by means of the circle in Fig. 36, Art. 84, by finding the tangential distance of I or J' from that circle, and drawing arcs with I and J' as centres and this tangential distance as radius. The points of intersection of these arcs will be the rotation centres, for (the tangent)$^2 = ID \,.\, J'D' = -\lambda$.

82. We will now consider the question

To construct on a straight line a row which shall be homographic to a row already described upon it.

We will give two methods of construction, and it will make the subject clearer if, instead of a single line, we imagine a double line consisting of two parallel lines L, L' indefinitely near to one another.

(1) Let abc be the characteristic of L, and take any three points a', b', c' to be the characteristic of L'. Art. 38.

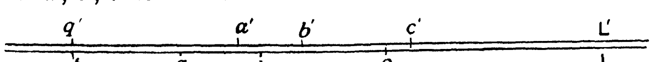

Fig. 34.

Take any point on the double line and let it be denoted by p when considered as a point on L, and by q' when considered as a point on L', and rotate the line L' about q' so as to make any angle with L, the points a', b', c' retaining their distances from q' (or p) unchanged. As we remarked in Art. 70, since

$$(abcm) = (a'b'c'm'),$$

the distances of corresponding points m, m' from p are independent of the angle between L and L'.

Then in Fig. 24, p. 39, by joining pairs of non-corresponding points (ab'), $(a'b)$ and (ac'), $(a'c)$, and drawing the line through their points of intersection γ, β, as in Art. 50, we obtain the cross-axis $p'q$, and the points (p, p') correspond, as do also (q, q'). By means of the cross-axis any number of pairs of points (m, m') can be found.

To find I and J', O and O'.

In Fig. 25, p. 41, let c, c' be any pair of corresponding points. Through c draw a line parallel to L' meeting $p'q$ in i. Then $c'i$ produced will meet L in the required point I. Through I draw a line parallel to $p'q$. This will meet L' in the required point J' by Art. 55.

Rotating L' about q' to its original position in the double line and bisecting IJ', we obtain the point O. To find O' let Oc' meet the cross-axis in ω. Then $c\omega$ will meet L' in the required point O'.

(2) Move one of the given lines L, L' parallel to itself until any pair of corresponding points, say (p, p'), coincide, and rotate the line L' through any angle about this common point, as in Fig. 26. By Art. 41 the ranges are now in perspective, centre S, and if we take any point m on L, and join Sm, it will cut L' in m', the point corresponding to m. Draw SI, SJ' parallels to L' and L, and rotate L' back again until it coincides with L, and then move it parallel to itself until p' is in its original position. We shall then have two homographic ranges on the given line, with the

points I, J' marked on it. It will be noticed that we have given two alternative methods of finding I and J', but in most problems the conditions will enable us to determine these points without having to use either of the methods, and then the easiest way of finding further pairs of points will be the method of Art. 84, as it involves no moving of the lines.

Methods of finding the common points e and f.

83. Since $Oe^2 = OJ' \cdot OO'$, Art. 73, Oe is a mean proportional between the two known lengths OJ' and OO', and can therefore at once be found by Euc. VI, 13.

We shall refer to this as Chasles' method of finding the common points (1852)*.

84. The following simple method of finding the common points and constructing any number of pairs of corresponding points on the two ranges is due to Prof. Alfred Lodge (1907)†. It requires the finding of I and J', but does not make use of O and O'.

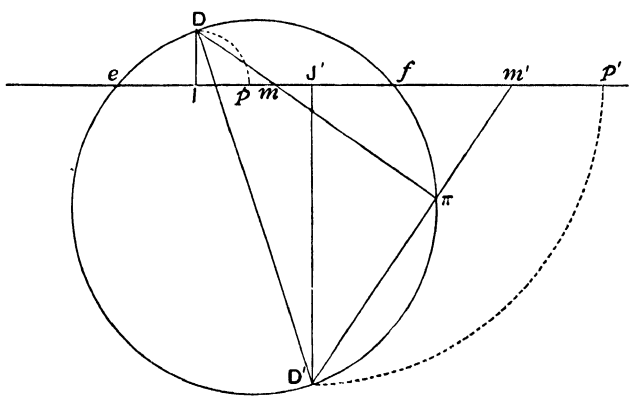

Fig. 35.

* *Traité de Géométrie Supérieure*, Art. 154.

† *Mathematical Gazette*, April, 1909.

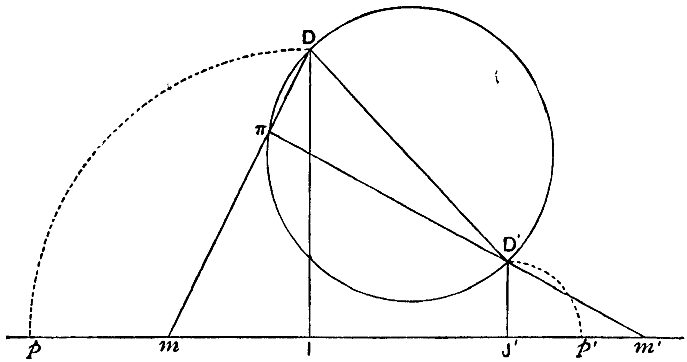

Fig. 36.

$$Ip \,.\, J'p' = \text{constant of correspondence}$$
$$= Ie \,.\, J'e$$
$$= Ie\,(Ie - IJ'),$$
$$\therefore\ Ie^2 - IJ' \,.\, Ie + Ip \,.\, p'J' = 0.$$

To solve this quadratic equation, graphically, measure from I and J' vertical distances $ID = Ip$, and $J'D' = p'J'$, ID and $J'D'$ being drawn on the same or opposite sides of IJ' according as Ip and $p'J'$ are measured in the same or opposite directions. Then the circle described on DD' as diameter will cut the given axis in the required common points e, f.

For if ID (produced if necessary) cuts the circle again in D_1,

$$Ie \,.\, J'e = eI \,.\, If = DI \,.\, ID_1 = DI \,.\, J'D' = Ip \,.\, J'p'.$$

The reason for putting the equation in the form

$$Ie^2 - IJ' \,.\, Ie + Ip \,.\, p'J' = 0$$

is to make it clear how the coefficients in any such quadratic equation are connected with the lines in the graph. The lengths and signs of the perpendiculars ID, $J'D'$ are the equivalents of the factors of the constant term, and the distance between them measured from the origin I is equal to the coefficient of Ie

with its sign changed, so that in Fig. 35, If is the positive root and Ie the negative root, if we consider IJ' as positive.

In Fig. 36, the roots are imaginary. In a vector sense they may be considered as represented by the rotation centres defined in Art. 81, since $OR^2 = -Oe^2$, and the roots are $IO \pm Oe$, that is, $IO \pm OR\sqrt{-1}$.

We shall refer to the above as Lodge's method.

Figs. 35 and 36 also give us a simple method of constructing any number of pairs of corresponding points on the two ranges.

For let m be any given point on the first range, and let Dm cut the circle in π. Join $D'\pi$ meeting the axis in m'. Then the triangles DIm, $m'J'D'$ are similar.

$$\therefore\ Im : ID = J'D' : J'm',$$

$$\therefore\ Im \,.\, J'm' = ID \,.\, J'D' = Ip \,.\, J'p'.$$

Therefore by Art. 65 the range m' is homographic to the range m. In Fig. 35 the common points are real, and in Fig. 36 they are imaginary. If the ranges are such that the circle on DD' touches the axis, the common points will coincide at the point of contact, as will also the points O and O'.

If we construct the Lodge circle for each pair of corresponding points we shall obtain a co-axial system since each passes through the points e, f. If the common points are imaginary, the circles have real limiting points which are the rotation centres, for the distance of each from the axis is equal to the tangential distance of O from any circle of the system.

Ex. If A, B are two fixed points on a circle, and P a variable point on it, and if PA, PB produced cut a fixed line (which does not cut the circle in real points) in M and M', shew that MM' subtends a constant angle at R, where R is either of the limiting points of the system of co-axial circles defined by the given circle and the given line as radical axis.

Find I on MM' such that the angle $AIM = APB$, *i.e.* so that A, P, M', I are concyclic. Then I is a fixed point, and

$$MI \,.\, MM' = MA \,.\, MP = (\text{tangent})^2 = MR^2.$$

Therefore the triangles MRM', MIR are similar, and the angle

$$MRM' = MIR,$$

which is fixed. Therefore the angle MRM' is constant.

85. In comparing the methods of Arts. 83 and 84 it will be noticed that they both depend upon first finding I and J', which it will be found in practice can generally be determined by inspection from the conditions of a problem. We thus obtain the characteristics $aI\infty$ and $a'\infty' J'$. Chasles' method then proceeds to find O', and the construction is completed by Euc. VI, 13. By Lodge's method it is not necessary to find O', and the amount of construction required is distinctly less. Moreover Lodge's circle, besides giving the common points, enables us to construct as many pairs of corresponding points as we please.

86. The following method given by Chasles* enables us to find the common points directly from the characteristics abc and $a'b'c'$, without finding I and J'.

Through any arbitrary point g describe two circles having ab' and ba' as chords, and intersecting in a second point g'. Through g describe two other circles having ac', ca' as chords, and intersecting again in g''. Then the circle round $gg'g''$ will intersect the given line in the common points required.

We may remark that this method, which depends on the construction of five circles, is one which it is not easy to use, as it is difficult in practice to construct the circles with sufficient accuracy to obtain more than approximate positions of the common points.

87. Another construction by means of a circle or conic will be given in Art. 160, at the end of Chapter XII.

NOTE. Art. 84 may be treated analytically as follows:

If the homographic equation of the two ranges is known to be of the second order, it can always be put in the form

$$(x_1 - a)(x_2 - b) + hk = 0,$$

where hk is never zero, *i.e.*, in the form

$$\frac{h}{x_1 - a} \cdot \frac{k}{x_2 - b} + 1 = 0.$$

* *Traité de Géom. Sup.*, Art. 263.

In Figs. 35 and 36, taking the common axis as the axis of x, suppose the point m is given $(x_1, 0)$. Find two points D, D' with coordinates (a, h) and (b, k) respectively, and describe a circle on DD' as diameter. From m, $(x_1, 0)$, draw mD meeting the circle again in π. Then $\pi D'$ will cut the common axis in m', $(x_2, 0)$, for

$$\text{the gradient of } mD \text{ is } \frac{h}{a - x_1},$$

$$\text{and the gradient of } m'D' \text{ is } \frac{k}{b - x_2},$$

and by the homographic equation their product $= -1$. Therefore mD and $m'D'$ meet on the circle.

Cor. 1. It follows that the common points are the points where the circle cuts the common axis.

Cor. 2. The point I is at $(a, 0)$ and J' at $(b, 0)$.

Cor. 3. The equation of the circle is

$$(x - a)(x - b) + (y - h)(y - k) = 0.$$

Therefore when $y = 0$, $(x - a)(x - b) + hk = 0$, giving the common points.

CHAPTER VIII

PROBLEMS OF THE THREE SECTIONS.—OTHER PROBLEMS WHOSE SOLUTIONS DEPEND ON FINDING THE COMMON POINTS OF TWO CO-AXIAL HOMOGRAPHIC RANGES

88. THE problem of finding the common points of two homographic ranges on the same axis is one of frequent occurrence, and can be applied to the solution of geometrical questions which in analysis would depend upon the solution of equations of the second degree, and we will therefore solve somewhat fully a few problems, which will enable the student to become familiar with the method. We will follow Chasles* in selecting for our purpose three of the most noted problems of the ancients, viz.:

(1) On Determinate Section. *τῆς διωρισμένης τομῆς.*

(2) On Spatial Section. *τῆς ἀποτομῆς τοῦ χωρίου.*

(3) On Proportional Section. *τῆς ἀποτομῆς τοῦ λόγου.*

These problems are of interest in themselves, for "the great geometer" Apollonius wrote separate treatises on them, intending them to be text-books on the application of analysis to geometry. They were extant in Greek in the time of Pappus, 400 A.D., and the fertility of the problems and the thoroughness with which they were treated may be inferred from his statement that the first treatise contained 83 propositions, the second 124, and the third 181. It was generally supposed that they had perished,

* *Traité de Géométrie Supérieure*, Chaps. XIV, XV.

either from the destroying hand of time, *edax rerum*, or from the still more destructive hands of barbarians, until Edward Bernard, who was Savilian Professor of Astronomy from 1673 to 1691, discovered an Arabic manuscript containing the section of ratio in the Bodleian Library. Being skilled in Oriental languages he began to translate it into Latin, but at his death his successor, D. Gregory, found that he had hardly completed a tenth part of it, and at the suggestion of Aldrich, Dean of Christchurch, the manuscript was submitted to his friend Halley, who had just been elected Savilian Professor of Geometry, and who took a keen interest in the editing of ancient mathematical writers. Undismayed by the fact that he did not know a word of Arabic, he cheerfully undertook to complete Bernard's unfinished task, and in the preface to the work he tells us how he did it. With the aid of that part of the manuscript which Bernard had translated (which consisted of only 13 pages out of 138), he first picked out those words whose meaning he was able to recognise from the context, and then by studying the argument and turning over and over in his mind what might be the meaning of the words which he did not recognise, by this method of deciphering he groped his way through nearly the whole of the book, and obtained a general idea of its contents. Then by recommencing and going over the same ground step by step again and again he managed to complete the work without the assistance of anyone else.

Having overcome this obstacle so successfully, it sounds almost hypercritical when he tells us that in addition to his other difficulties the manuscript was badly written, the diacritical points were wanting from many of the letters, and occasionally words, and even sentences, were missing, so that, as he says, it required a soothsayer rather than an interpreter to divine the true meaning. He then proceeded to restore the treatise *de Sectione Spatii*, and here he had not even the help of an Arabic version, but merely a short description of its contents given by

Pappus in the preface to his seventh book of the *Col. Math.* together with a few Lemmas dealing with the subject. The two treatises were published by him in Latin in one volume in 1706.

The treatise on determinate section was restored from similar scanty materials by Snell in 1601, by Lawson in 1772, by Wales in 1772, and by R. Simson in his posthumous works published in 1776.

The road which all these writers have taken is, however, a long and toilsome one, and as Chasles points out, homography gives us a simple method by which we can solve the problems either in their most general form, or in any of the particular cases which they can assume.

I. Determinate Section.

89. *Given four collinear points a, a', b, b', it is required to find another point m, collinear with them, such that*

$$\frac{ma \,.\, mb'}{mb \,.\, ma'} = \mu \text{ (const.)} = \frac{pq}{qr}.$$

There will evidently be two points satisfying the given conditions, for the equation is a quadratic, and these two points are obviously the common points of the two homographic divisions formed by $\dfrac{ma}{mb} = \mu \,.\, \dfrac{m'a'}{m'b'}$, where a, b are two points in the first row, and a', b' their correspondents in the second. See Art. 62.

To determine the points I and J'.

Let m' be at ∞'. Then $\dfrac{Ia}{Ib} = \mu$.

Let m be at ∞. Then $\dfrac{J'b'}{J'a'} = \mu$.

Having found I and J', we can determine the common points e, f by any of the methods given above.

Chasles' method, Fig. 37.

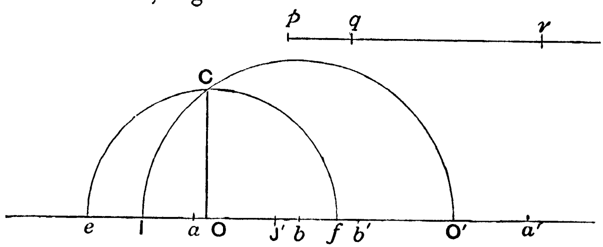

Fig. 37.

Bisect IJ' in O, and considering O as a point in the first row, find its corresponding point O' from either of the relations

$$\frac{aO}{bO} = \mu \frac{a'O'}{b'O'}, \text{ or } IO \, . \, J'O' = Ia \, . \, J'a'.$$

On IO' describe a semi-circle, in which draw OC perpendicular to IO'. Then the circle with centre O and radius OC will cut the axis in the required points e, f.

Lodge's method, Fig. 38.

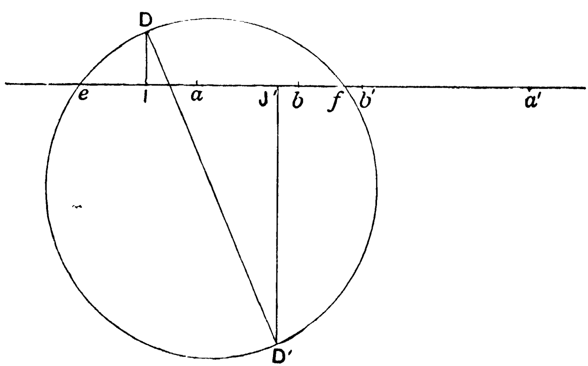

Fig. 38.

$$Ie \, . \, J'e = Ia \, . \, J'a',$$

$$\therefore \; Ie\,(Ie - IJ') + Ia \, . \, a'J' = 0,$$

$$\therefore \; Ie^2 - IJ' \, . \, Ie + Ia \, . \, a'J' = 0.$$

On perpendiculars through I, J' take $ID = Ia$, and $J'D' = a'J'$, measuring them on opposite sides of IJ', because Ia and $a'J'$ are of opposite signs. Then the circle described on DD' as diameter will cut the given line in the required common points e, f.

II. Spatial Section.

90. *Given two straight lines AL, BL' on which A and B are fixed points, it is required to draw through a fixed point P a transversal ePe' forming on AL, BL' the two segments Ae, Be' such that $Ae \cdot Be' =$ a given quantity K.*

Let a, a' be two points on AL, BL' such that $Aa \cdot Ba' = K$. Then in their different positions the divisions (a) and (a') are homographic by Art. 65; and if $a'P$ meets AL in α', the ranges (a) and (α') are homographic by Art. 39, and we have to find the common points of these ranges.

When a is at infinity, a' is at B, and α' at J', the point where BP meets AL.

When α' is at infinity, a' is at I', where PI' is parallel to AL, and I is obtained from the relation $AI \cdot BI' = K$.

[The position of I can also be found as follows. If in Fig. 40 we rotate the line L' about I' so as to come into the position $I'P$, and if B, a', ... come into the positions B_1, a_1', ..., then AB_1 is the cross-axis of the ranges A, a, ..., B_1, a_1', ... Therefore if we join $I'a$ cutting AB_1 in X, $a_1'X$ will pass through I. Similarly we can find Ω' in Fig. 39.]

Employing Chasles' method, Fig. 39, find O the mid-point of IJ', and Ω' its corresponding point on BL' from the relation $AO \cdot B\Omega' = K$, and join $\Omega'P$ meeting AL in O'. Describe the semicircle on IO' and obtain the common points e, f as in the preceding problem.

Join eP, fP and produce them to meet BL' in e', f'. Then the two pairs of points e, e' and f, f' are the points required.

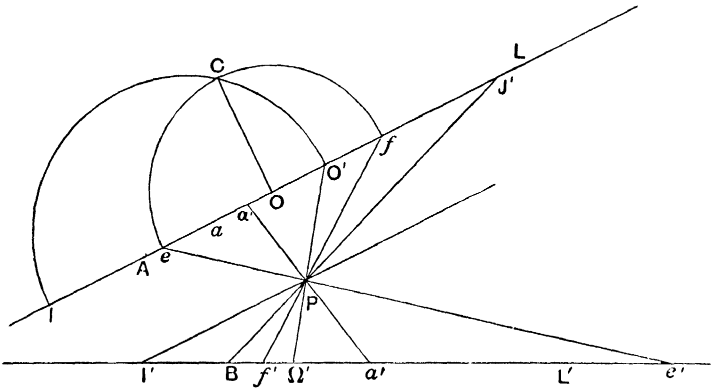

Fig. 39.

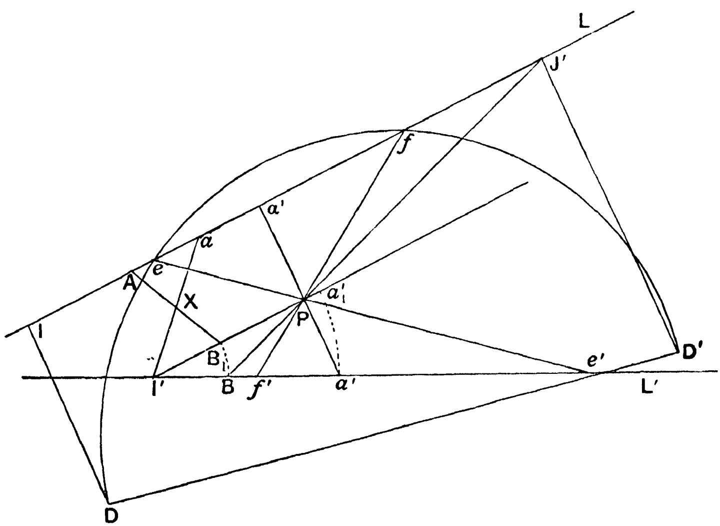

Fig. 40.

Employing Lodge's method, Fig. 40, as in Art. 89, we have

$$Ie^2 - IJ' \,.\, Ie + Ia \,.\, a'J' = 0.$$

Find the points D, D', which will be on the same side of IJ', because Ia and $a'J'$, the factors of the last term, are both measured in the same direction. Then the circle on DD' as diameter will cut AL in the points e, f, and the problem can be completed as before.

III. Proportional Section.

91. *Given two straight lines AL, BL' on which A, B are fixed points, it is required to draw through a fixed point P a transversal ePe' forming on AL, BL' the two segments Ae, Be' which shall be in a given ratio λ.*

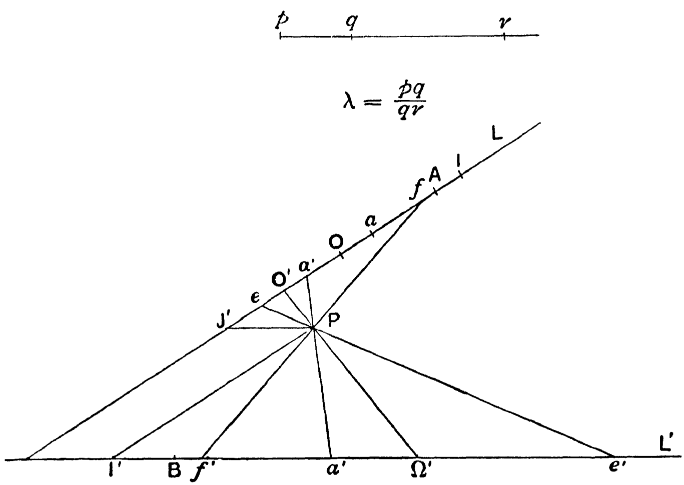

Fig. 41.

Let a, a' be two points on AL, BL' such that $\dfrac{Aa}{Ba'} = \lambda$. Then in their different positions the divisions (a) and (a') are homographic by Art. 67; and if $a'P$ meets AL in α', the ranges (a) and (α') are also homographic, and their common points will give us the points required.

To find I, the position of a when a' is at infinity, draw PI' parallel to AL, and take $\frac{AI}{BI'} = \lambda$, for a' then coincides with I'.

To find J', the position of a' when a is at infinity, and when a' is also at infinity by Art. 67, draw PJ' parallel to BL'.

If we wish to use Chasles' method, find O the mid-point of IJ'. On BL' find Ω' corresponding to O from the relation $\frac{AO}{B\Omega'} = \lambda$, and join $\Omega'P$ meeting AL in O', and complete the construction as in the problems of Arts. 89, 90.

Employing Lodge's method we have

$$Ie^2 - IJ' \cdot Ie + Ia \cdot a'J' = 0.$$

The points D, D' must now be taken on the same side of IJ', since Ia and $a'J'$ have the same sign.

It has been left to the student to draw the circles according as he employs Chasles' or Lodge's method.

92. *Given a triangle ABC, and three points P, Q, R in its plane, it is required to inscribe in ABC another triangle whose sides shall pass through P, Q, R.*

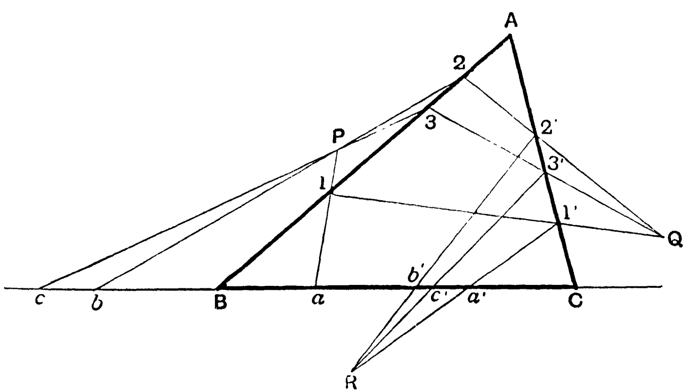

Fig. 42.

Through P draw any straight line $P1a$, cutting BC in a and AB in 1. Join $Q1$ cutting AC in $1'$. Join $R1'$ cutting BC in a'.

In the same way we can find as many pairs of points such as a, a' as we please on BC, the ranges (a) and (a') will be homographic, and their common points will give us vertices of the two triangles which can be constructed satisfying the given conditions.

The process of finding any pair of corresponding points such as a, a' is what Chasles calls a *construction d'essai*, and if the points a, a' happened to coincide, then we should have found one of the common points*, but if not, the segment aa' would be a measure of the error, and, as he puts it, we may make use of three similar errors given by three trial constructions to solve the problem, so that there is a sort of analogy between this general method and the arithmetical *rules of false position.* It will be noticed, however, that all that we obtain from such constructions in general is three pairs of corresponding points, which constitute what we have called the characteristics of the two homographic ranges, and even if we chanced to hit upon one pair of corresponding points which happened to coincide, it would not help us at all to find the other pair.

In Fig. 42 let a, a'; b, b'; c, c' be three pairs of corresponding points obtained as above. Then abc and $a'b'c'$ are the characteristics of the ranges, and we might now proceed, as in Art. 82, to rotate one of the ranges, construct the cross-axis and find I, J', &c. It will be found in practice, however, that this is a long process requiring a considerable amount of construction, which increases the chance of error in the result, and either I or J' can be obtained much more easily and accurately by direct use of the fact that it is the position which one of the points a, a' assumes when the other is at infinity; in fact, we use the special characteristics $aI\infty$, $a'\infty'J'$ instead of the general ones abc, $a'b'c'$.

To find I, suppose a' at infinity, and let the line through R

* We have a similar *construction d'essai* in Euc. II, 14; VI, 28; XI, 11.

parallel to BC meet CA in $2'$. Draw $2'Q$ meeting AB in 2. Then $2P$ will meet BC in I.

To find J', let the line through P parallel to BC meet AB in 1. Draw $1Q$ meeting AC in $1'$. Then $1'R$ will meet BC in J'.

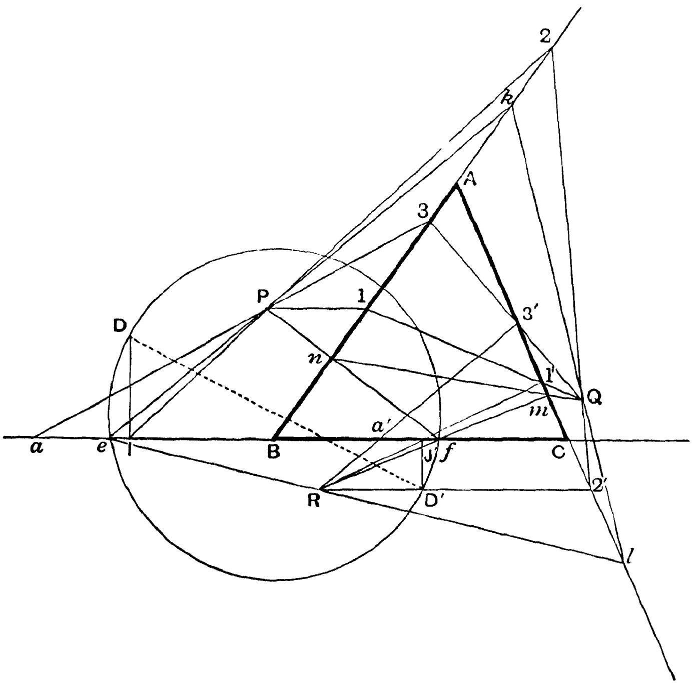

Fig. 43.

In Fig. 43 we have employed Lodge's method, taking the points D, D' on opposite sides of IJ', because Ia, $a'J'$ are of opposite signs. The circle on DD' as diameter intersects BC in the common points e, f. Join eP meeting BA in k. Draw kQ meeting AC in l. Then le will pass through R, and ekl is one of the required triangles. Similarly, starting from the point f, we obtain the triangle fnm, which also satisfies the given conditions.

93. In the next two examples we will merely indicate the opening steps by which the points I, J' can be obtained, leaving it to the student to draw the figures and to find the common points by one of the methods given above. Questions of this class might also be solved by finding the characteristics, and then projecting them on a circle or conic, as will be explained in a later chapter. This is the method employed by Cremona in his *Éléments de Géométrie Projective* (1875), and in Section XIX of that work several of these problems are treated in this manner. It should, however, be impressed upon a student that, in dealing with any given problem, when he has obtained the two homographic ranges on a line, and found the three pairs of corresponding points, it is not sufficient for him to say "Hence the common points of these ranges which satisfy the conditions of the problem can be found." For a complete solution the actual positions of the common points ought to be determined in every case.

94. *Given two homographic ranges (a) and (a') on two lines AL, $A'L'$, and two points P, P' in the same plane with them, it is required to find two corresponding points e, e' in the ranges such that the lines Pe and $P'e'$ will contain a given angle ϕ.*

Since the ranges are given, if we take any point in the one, we are supposed to know the position of its corresponding point in the other. Let a, a' be any pair of corresponding points. Join $P'a'$, and through P draw a line making the given angle ϕ with $P'a'$ and meeting AL in α'. Then the ranges (a') and (α') are homographic, as are also (a) and (a'), and consequently by Art. 39 the ranges (a) and (α') are also homographic and we evidently have to find their common points.

To find the point I in the range (a) corresponding to the point at infinity in (α'). When α' is at infinity, $P\alpha'$ is parallel to AL. Through P' draw a line making an angle ϕ with AL, and meeting $A'L'$ in I'. Then the point I in the range (a) corresponding to I' in (a') is the point required.

To find J', suppose a is at infinity, and let j' be the corresponding point in (a'). Join $j'P'$, and through P draw a line making with $j'P'$ the angle ϕ. This will meet AL in the required point J', &c.

95. *Given two straight lines L, L', it is required to find on them two points a, a' such that the line aa' will subtend given angles ϕ, ϕ' at two fixed points P, P'.*

On L' take any point A_1 and make the angle $A_1PA = \phi$, and $A_1P'a' = \phi'$, the points A and a' being on the line L. Then as A_1 moves along L', A_1 and a' trace out homographic ranges, as do A_1 and A, and, by Art. 39, A and a'. Hence we have to find the common points of the ranges (A) and (a').

To find I. When a' is at infinity, $P'a'$ is parallel to L, A_1 is at A_2 such that $A_2P'a' = \phi'$, and A is at I, where $A_2PI = \phi$.

To find J'. When A is at infinity, PA is parallel to L, A_1 is at A_3 such that $A_3PA = \phi$, and a' is at J', where $A_3P'J' = \phi'$, &c.

EXAMPLES.

1. Determine on a given line a segment which shall subtend given angles at two given points.

2. Determine on a given line a segment of given length which shall subtend a given angle at a given point.

3. AL, AL' are two given lines. P is a given point in their plane, and a a given point in AL. Through P it is required to draw a transversal Pbb' meeting AL in b and AL' in b' such that $Ab' = ab$.

4. In a given triangle inscribe a rectangle equal to a given square.

5. Given a plane polygon of any number of sides, and the same number of points in its plane, inscribe in the polygon another polygon whose sides will pass through the given points.

6. A, B are two fixed points in a given straight line. It is required to find in the straight line two other points E, F so that EF may be of given length, and the cross-ratio $(ABEF)$ of given magnitude.

7. Given two homographic ranges on two lines AL, $A'L'$, and two others on BM, $B'M'$, it is required to draw through a given point P two straight lines each of which will intersect AL, $A'L'$ in a pair of corresponding points, and will do the same with BM, $B'M'$.

[Let a, a' be a pair of corresponding points on AL, $A'L'$, and let Pa meet BM in b, and let b' be the point on $B'M'$ corresponding to b. Then if Pb' meets $A'L'$ in α', the ranges (a') and (α') are homographic, and the lines joining P to their common points are the two lines required. Find I, J', &c.]

8. Given a triangle ABC and P a fixed point in its plane, draw through P a line cutting AC in m and BC in n such that the triangle mnC may be equal to the triangle ABC.

9. Given a triangle ABC and a point P on the parallel to BC through A, lying between A and the median through B (produced). Draw through P a line which will bisect the triangle ABC.

For other examples of this class the student is referred to

Chasles, *Géom. Sup.* pp. 219—223.
Cremona, *Géom. Proj.* (1875), pp. 179—188.
Townsend, *Mod. Geom.* (1863), vol. II, pp. 257—275.

CHAPTER IX

INVOLUTION

96. DEF. When two co-axial homographic ranges have the points I and J' coincident, the two ranges are said to form a *range in involution* or an *involution range*. Hence

Range in Involution.

A system in involution consists of two co-axial homographic ranges, all that is necessary being that they should be placed so that I and J' coincide.

We must remember that homographic ranges are of two kinds. We have (1) those in which I and J' are at a finite distance. These are homographic ranges of the second order, and are by far the most important, and the involution to which we devote our chief attention is when I and J' of such ranges coincide. But besides these there are (2) homographic ranges of the first order (Arts. 67, 68), when the ranges are divided similarly, and both I and J' are at infinity. We shall deal with the condition that such ranges shall be in involution in Art. 107. But for the present, and in general, we shall confine our attention to ranges of the second order, and proceed to discover what special properties these possess, when they are in involution. One of the most important of these, leading indeed to a second definition of involution, follows from consideration of the construction in Art. 82, where we may of course select any point we please on the double line for the point of rotation, and the two points on L and L' corresponding to the point which we select on the double line will in general be separate, and the cross-axis which

we obtain by joining them after rotation will have a different position for each point selected on the double line, but *it will always be parallel to the line joining IJ'*, Art. 55.

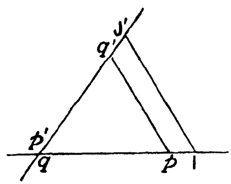

Fig. 44.

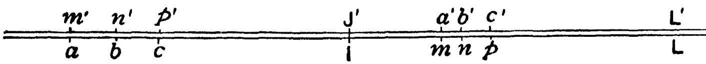

Fig. 45.

Suppose now we have the double line with its two homographic ranges, and we slide the line L' along the double line until the point J' coincides with I, and then rotate L' about any point $p'\,(=q)$ through any angle.

Let pq' be the cross-axis. Then by Art. 55 pq' is parallel to IJ'; and since $p'J' = qI$, therefore $p'q' = qp$, and if L' is rotated back again about p' until it coincides with the double line, the point q' will fall upon the point p, and therefore the point p' (or q) has the same correspondent whether it is considered to be a point on L or on L'; and as $p'\,(=q)$ may be any point on the double line, we may say that

When two co-axial homographic ranges have the points I and J' coincident, then every point on the axis has its two corresponding points coincident.

This is also clear from consideration of the equation

$$Ip \,.\, J'p' = \text{const.},$$

for if J' coincides with I, and therefore also with their mid-point O we have $Op \,.\, Op' = \text{const.} = Oe^2 = Of^2$, and the symmetry of the

relation shews that no matter which range p belongs to, p' is its correspondent. The converse is also true; viz. that if any point taken on the axis (other than one of the common points) has its two corresponding points coincident, the points I, J' will coincide. This follows at once from Fig. 44, since if $p'q' = pq$, we must also have $p'J' = qI$, and therefore when the lines are rotated so that q' falls on p, J' must fall on I. Consequently we have a second definition of involution, co-extensive with the first, involving the same geometrical fact, though emphasising another property, viz.

Ranges in Involution.

DEF. Two co-axial homographic ranges are in involution when any point on the axis (other than one of the common points) has the same corresponding point, whether the given point belongs to the first range or to the second. This property is given in symbolic form in Art. 98.

The common points must be excluded from this condition as each of them corresponds to itself whether the ranges are in involution or not. The point taken on the axis will have its correspondents coinciding with each other, but not with itself. The sufficiency of the test, and the necessity of excluding the common points, are both obvious from Fig. 44.

Conjugate Points.

Double Points.

97. DEF. Since when two co-axial ranges are in involution their pairs of corresponding points are interchangeable, we will in this case call them *conjugate* points, and the common points of the ranges we will call *double* points. This distinction will serve to shew the reader whether we are speaking of two co-axial homographic ranges in general, or in the special case where they form a system in involution.

Since I, J' coincide, they also coincide with their mid-point O.

Centre.

DEF. The point O is now called the *centre* of the system*.

* Chasles, *Aperçu Historique*, p. 318.

When two ranges form a system in involution we will denote it by $(aa', bb', cc' \ldots)$, where $aa', bb', cc' \ldots$ are pairs of conjugate points, and we shall in general use the letters e, f in speaking of the double points.

98. When we have a system in involution, if we take three pairs of conjugate points and consider four of the six points (provided they do not form two pairs of conjugates), it is easy to see that their cross-ratio is equal to that of their conjugates*.

Thus if a, a'; b, b'; c, c' are the three pairs of points, we may take a, b, c, a' as belonging to the first range, and then by the secondary definition of involution given in Art. 96 it follows that a', b', c', a are their correspondents in the second range, so that $(abca') = (a'b'c'a)$, or referring to Fig. 45,

$$\begin{aligned}(abca') &= (abcm) \text{ since } m \text{ and } a' \text{ coincide}\\ &= (a'b'c'm') \text{ since the rows are homographic}\\ &= (a'b'c'a) \text{ since } m' \text{ and } a \text{ coincide.}\end{aligned}$$

This property is the symbolic form of the definition of Art. 96, *ad fin.* which established two converse theorems, viz. that if a, a'; b, b'; c, c' are in involution, then $(abca') = (a'b'c'a)$, and similarly for other sets (see equations 1—7, in Art. 106), and conversely, if

$$(abca') = (a'b'c'a),$$

then a, a'; b, b'; c, c' are pairs of points in involution. It will be found that this property is in many problems the easiest involution property to discover. See also Art. 105.

99. We now come to a third property of involution, deducible at once from Art. 98. Since $(aa'ef) = (a'aef)$, the range $aa'ef$ by Art. 28 is harmonic, and *the double points are harmonic conjugates for each pair of conjugate points.*

* Chasles, *Aperçu Historique*, p. 313.

As a special case *if one of the double points is at infinity, the other bisects each of the segments joining the pairs of conjugates.* When this case occurs, the homography is of the first order, and is discussed in Art. 107.

100. The property given in Arts. 96, 98 may also be seen in Fig. 46 where the ranges form a system in involution, centre O. Small letters denote points in the upper range, and their conjugates are denoted by accented capitals, so that (a, A') represent a pair of conjugates, as do also (a_1, A_1').

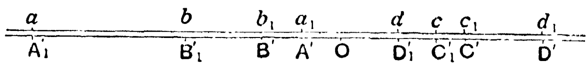

Fig. 46.

The relations $Oe^2 = Of'^2 = Oa \,.\, OA' = Oa_1 \,.\, OA_1' = \ldots$ shew that as regards any pair of conjugates it is immaterial which of them we assign to the upper, and which to the lower range, so that we can either say $Oa \,.\, OA'$ or $OA_1' \,.\, Oa_1$; *i.e.* any point has the same conjugate whether it belongs to the upper or the lower range, and any pair of conjugate points (a, A') give rise to another pair of conjugates (a_1, A_1'), which coincide with the former pair when taken inversely.

The same result can be deduced from the following proposition.

101. *In two equicross ranges, if (a, A') are any pair of corresponding points, we can always find another pair of corresponding points (a_1, A_1') such that the segments aa_1 and $A'A_1'$ are of equal length.*

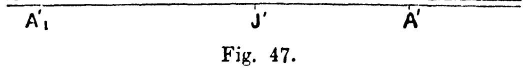

Fig. 47.

For on the upper range take a point a_1 such that $Ia_1 = J'A'$. Then the correspondent of a_1 is A_1', where

$$Ia_1 \,.\, J'A_1' = Ia \,.\, J'A',$$

$$\therefore \; J'A_1' = Ia,$$

$$\therefore \; aI + Ia_1 = A_1'J' + J'A', \quad i.e. \;\; aa_1 = A_1'A'.$$

Also $$aA_1' = a_1A' = IJ'.$$

Hence if we are given any two co-axial homographic rows, and if we move one of them (say the accented row) along the other until the points I, J' coincide at O, a will coincide with A_1', and a_1 with A', and as a is any point on the axis, we have the property of the previous Article.

102. *Given two pairs of conjugate points a, a'; b, b'; to find the centre of the involution.*

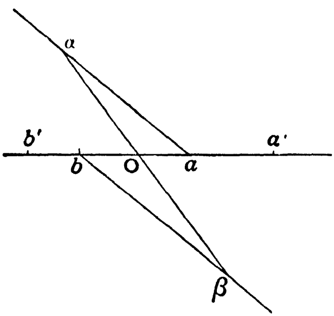

Fig. 48.

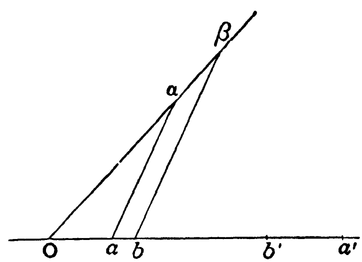

Fig. 49.

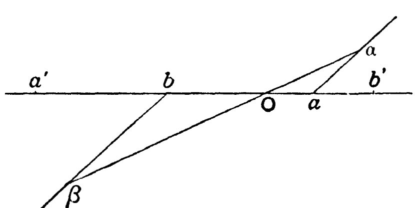

Fig. 50.

From the relation $Oa \,.\, Oa' = Ob \,.\, Ob'$ we have

$$\frac{Oa}{Ob} = \frac{Ob'}{Oa'} = \frac{Oa + ab'}{Ob + ba'} = \frac{ab'}{ba'}.$$

Through a and b draw any pair of parallel lines $a\alpha$, $b\beta$ in the same or opposite senses, according as ab', ba' are in the same or opposite directions, and take $a\alpha = ab'$, and $b\beta = ba'$. Then the line joining $\alpha\beta$ will meet the axis in the required point O.

103. The value of the expression $Oa \,.\, Oa'$ is called the *power of the involution*, and is easily found in terms of the segments between the points a, a', b, b'.

For, as in Art. 102,

$$Oa = \frac{ab}{b'a'} \,.\, Ob' = \frac{ab}{b'a'}(Oa + ab'),$$

$$\therefore\ Oa\,(b'a' - ab) = ab \,.\, ab'.$$

Again,

$$Oa' = \frac{b'a'}{ab} \,.\, Ob = \frac{b'a'}{ab}(Oa' + a'b),$$

$$\therefore\ Oa'\,(ab - b'a') = b'a' \,.\, a'b,$$

$$\therefore\ Oa \,.\, Oa' = \frac{ab \,.\, ab' \,.\, a'b \,.\, a'b'}{(ab + a'b')^2}.$$

104. Just as three pairs of points are sufficient to determine the homography of two ranges, two pairs of points are sufficient to determine an involution, and may be called its characteristic. For if a, a'; b, b' are two given pairs of points, we can by Art. 102 find the point O, and the involution condition will then determine the conjugate c' of any point c, which we can find by Euc. VI, 12, from the relation $Oc \,.\, Oc' = Oa \,.\, Oa'$. Therefore if six points are in involution there is a relation between them from which, if five are given, the sixth can be found; and three pairs is the least number between which this relation can exist. It must be remembered that since two pairs determine the involution, if a, a'; b, b'; c, c' are in involution, and also a, a'; b, b'; d, d', it follows that a, a'; b, b'; c, c'; d, d' form a system in involution.

105. If we have given six collinear points which are connected by an equation of cross-ratios, the following consideration enables us to determine by inspection of the equation whether the points are in involution, or not.

If from six points p, q, r, s, t, u we can form two equicross ranges, such that of the two points (say q, u) which are necessarily common to both ranges each has the other for its correspondent wherever they occur, the six points are in involution. Thus, if $(pqru) = (tusq)$, then p, t; q, u and r, s are three pairs of points in involution.

Of course we may have to rearrange one of the cross-ratios before it takes the requisite form, but if we can by interchanging pairs of letters in accordance with the rule of Art. 3 put one of the members into the requisite form whilst retaining the equality of the cross-ratio, there is involution, *e.g.* Suppose we are given $(pqru) = (utqs)$. The right-hand side can be written in the form $(tusq)$, and therefore $(pqru) = (tusq)$, shewing that there is involution since the repeated letters now correspond. If we cannot do this, there is not involution, as in $(pqru) = (qust)$ or $= (stqu)$.

The necessary and sufficient condition for involution is the following rule:

Whatever places the repeated letters occupy in one of the cross-ratios, both or neither of them must occupy these places in the second cross-ratio.

Important Involution Equations.

106. Since $aa'ef$ is a harmonic range,

$$\therefore \ (aa'ef) = -1,$$

$$\therefore \ \frac{ae}{af} = -\frac{a'e}{a'f},$$

and referring this to any arbitrary origin m,

$$\frac{me - ma}{mf - ma} = -\frac{me - ma'}{mf - ma'},$$

$$\therefore \ (ma + ma')(me + mf) = 2ma \,.\, ma' + 2me \,.\, mf,$$

and if O, α are the mid-points of ef, aa',

$$2m\alpha \,.\, mO = ma \,.\, ma' + me \,.\, mf.$$

Similarly if β is the mid-point of bb',

$$2m\beta \,.\, mO = mb \,.\, mb' + me \,.\, mf,$$

$$\therefore\ 2\alpha\beta \,.\, mO = mb \,.\, mb' - ma \,.\, ma' \,*,$$

with two similar equations obtained by introducing γ the mid-point of cc'.

If m coincides first with a, and then with a', we obtain

(1) $2\alpha\beta \,.\, aO = ab \,.\, ab'\,\dagger$, (2) $2\alpha\beta \,.\, a'O = a'b \,.\, a'b'$.

Similarly we have

(1) $2\alpha\gamma \,.\, aO = ac \,.\, ac'$, (2) $2\alpha\gamma \,.\, a'O = a'c \,.\, a'c'$,

$$\therefore\ \frac{\alpha\beta}{\alpha\gamma} = \frac{ab \,.\, ab'}{ac \,.\, ac'} = \frac{a'b \,.\, a'b'}{a'c \,.\, a'c'}, \quad \text{i.e. } (aa'bc) = (a'ab'c') \;\ldots(1).$$

Similarly

$$\frac{\beta\gamma}{\beta\alpha} = \frac{bc \,.\, bc'}{ba \,.\, ba'} = \frac{b'c \,.\, b'c'}{b'a \,.\, b'a'}, \quad \text{i.e. } (bb'ca) = (b'bc'a') \;\ldots(2),$$

and

$$\frac{\gamma\alpha}{\gamma\beta} = \frac{ca \,.\, ca'}{cb \,.\, cb'} = \frac{c'a \,.\, c'a'}{c'b \,.\, c'b'}, \quad \text{i.e. } (cc'ab) = (c'ca'b') \;\ldots(3).$$

By suitable multiplication we may obtain the properties

$$ab' \,.\, bc' \,.\, ca' = - a'b \,.\, b'c \,.\, c'a \,\ddagger \text{ or } (abc'a') = (a'b'ca) \;\ldots(4),$$

$$ab' \,.\, bc \,.\, c'a' = - a'b \,.\, b'c' \,.\, ca \text{ or } (abca') = (a'b'c'a) \;\ldots(5),$$

$$ab \,.\, b'c' \,.\, ca' = - a'b' \,.\, bc \,.\, c'a \text{ or } (ab'c'a') = (a'bca) \;\ldots(6),$$

$$ab \,.\, b'c \,.\, c'a' = - a'b' \,.\, bc' \,.\, ca \text{ or } (ab'ca') = (a'bc'a) \;\ldots(7).$$

Any one of these seven equations expresses the condition that must hold when the six points a, a'; b, b'; c, c' are in involution, and consequently from it each of the other six equations can be obtained. These results of course follow directly from the definitions of Art. 96. The object of giving them here is to call attention to the fact that the principles involved in the

* Pappus, Bk VII, Props. 45—56. † Pappus, Bk VII, Prop. 41.

‡ Pappus, Bk VII, Prop. 130. See also Appendix I.

definitions of involution can be just as readily obtained from properties given by Pappus.

107. *Centre of the involution at infinity.*

If we consider the distance between the double points e, f to gradually increase, so that, while e remains at a finite distance f recedes to infinity, the centre O of the involution, which by Art. 72 is midway between them, is also at infinity. As this point O is formed by the coincidence of I and J', it follows that the ranges are similar, but we shall see that the involution condition of Art. 99 makes them not only similar, but identical, in opposite senses. For when f is at infinity, e bisects each of the segments aa', bb', ... Hence $ea = -ea'$, $eb = -eb'$, ...,

$$\therefore\ ab = -a'b', \text{ and similarly } bc = -b'c', \text{ \&c.}$$

Consequently, when the centre of the involution is at infinity, and one of the double points is at a finite distance, while the other is at infinity, the two ranges are identical, but in opposite senses, and the finite double point bisects each of the segments joining pairs of conjugate points.

This is the only practical case of involution of the first order, for if both double points were at infinity, every point on the range would have its conjugate at infinity. Consequently, similar ranges cannot be in involution unless they are identical, and in opposite senses, and then they are always in involution, for on drawing a figure it will be seen that the point which bisects the segment joining one pair of conjugate points necessarily bisects every other pair.

To avoid repetition in future we would remark that in all cases of involution the ranges are supposed, unless it is otherwise stated, to have their points I, J' coincident at a finite distance, and consequently their homographic relation is of the second order.

108. Since $Oa \,.\, Oa' = Ob \,.\, Ob' = \ldots = Oe^2 = Of^2$,

it is evident that if any pair of conjugate points such as a, a' are on opposite sides of O, the product $Oa \,.\, Oa'$ is negative, and

therefore the double points e, f are imaginary. When this is the case for any pair of conjugates, of course it must hold for every pair, for if the product of any pair $Oa \,.\, Oa'$ is negative, that of every pair must also be negative.

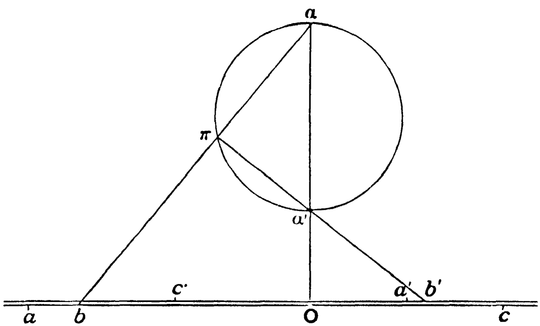

Fig. 51.

A little consideration of Fig. 51 will shew that when the point O divides the segments aa', bb' internally, if $Oa > Ob$, then $Oa' < Ob'$, and therefore the segments aa', bb' overlap, and this will obviously be the case with all the segments joining pairs of conjugates, and the double points are imaginary.

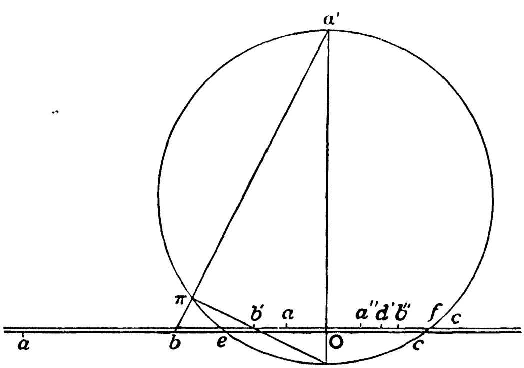

Fig. 52.

On the other hand the double points will be real when the product of any pair such as $Oa \,.\, Oa'$ is positive. The different positions where this can be the case are shewn in Fig. 52, in which, if we consider any pair of the segments aa', bb', cc', dd' we see that any segment lies entirely within or entirely without the other, and that no two segments overlap. Hence we have the simple rule:

The double points are real or imaginary according as the segments joining pairs of conjugate points do not or do overlap.

This might also be shewn by using Lodge's method, Art. 84. Thus in Figs. 51, 52, if we draw Oa, Oa' perpendicular to the axis equal to Oa, Oa' respectively, and on opposite sides of the base if the product $Oa \,.\, Oa'$ is positive, and on the same side if it is negative, the double points will be the intersection of the line and circle.

Lodge's circles also give us a simple method of constructing the conjugate of any point m on the base. For we have merely to join ma cutting the circle in M. Then $a'M$ will meet the base in the required point m'. The construction holds whether the double points are real or imaginary.

109. If we rotate the accented row about the point O through two right angles it will still be homographic to the other, and will form with it another system in involution; and since each of its points of division will now be on the opposite side of O to what it was before, we see that if the first system is an overlapping one, the second is a non-overlapping one, and *vice versa.*

One-to-One Correspondence.

110. If we take the general relation given in Art. 70 and express the condition that I and J' should both coincide with O, we have

$$pm \,.\, pm' - pO\,(pm + pm') + pO \,.\, pp' = 0,$$

which is of the form

$$xx' + h(x + x') + l = 0.$$

This gives us a one-to-one correspondence in which x and x' are interchangeable, and is merely another way of expressing the fact that the two series of points given by the equation have the property mentioned in Art. 96 and are in involution.

If the homographic equation is of the first order, if x and x' are to be interchangeable, it must be of the form

$$x + x' = K,$$

in which case one of the double points is at infinity, and the other at a distance $\frac{K}{2}$ from the common origin.

Involution Pencils.

111. Def. If the divisions of an involution range are joined to an external point, the pencil so formed is called an *involution pencil.*

Involution Pencil.

When two concentric pencils form a system in involution, we will denote it by $V(aa', bb', \ldots)$, where Va, Va' are a pair of conjugate rays, and Ve, Vf will in general be used to indicate the double rays.

By Art. 21 any transversal will cut an involution pencil in an involution range, and the double rays of the pencil will cut the transversal in the double points of the range.

By Art. 31 the angle between any pair of conjugate rays is divided harmonically by the double rays; and conversely, if aVa', bVb', cVc' are three angles which are all divided harmonically by the same pair of lines Ve, Vf, then $V(aa', bb', cc')$ is an involution pencil having Ve and Vf for double rays.

It should be noticed that there is no ray which can be called the central ray of the pencil, and in that respect it differs from an involution range. The ray conjugate to that drawn parallel

to the range passes through the centre of the range, and of course this ray will be different for different transversals, except when the transversals are parallel.

112. *In an involution pencil there exists one, and in general only one, pair of conjugate rays at right angles. When there is more than one, every pair of conjugate rays intersect at right angles.*

Let the pencil, vertex P, cut any transversal in the pairs of conjugate points a, a'; b, b'; ..., and let O be the centre of the involution range on the transversal. Join PO, and on PO, or PO produced, take a point Q such that $PO \,.\, OQ = aO \,.\, Oa' = \ldots$.

Then every circle through the two points P, Q will cut the transversal in a pair of conjugate points. There will be in general one and only one such circle having its centre in the transversal; and this alone will cut it in two conjugate points c, c' which will subtend a right angle at P. If, however, $PO = OQ$, and PQ cuts the transversal at right angles, every such circle will have its centre in the transversal, and all pairs of conjugate rays will be at right angles.

It follows from the above that

If any number of right angles have the same vertex, their sides form an involution pencil.

Such a pencil may be called *orthogonal.*

Circular Points at Infinity.

113. Since the involution range formed by an orthogonal pencil on any transversal is overlapping, the double rays of an orthogonal pencil are imaginary. If the rays of the pencil are produced to meet the line at infinity, they determine on it an involution range of ideal points with imaginary double points. These double points, though imaginary, are of very great import-

ance in connection with the subject. They are in a sense unique. For any two orthogonal systems each containing an infinite number of rays are superposable by mere translation without rotation, and parallel rays of the two systems will correspond to each other. These parallel rays intersect in the line at infinity, as do also their corresponding double rays, so that we may consider that all orthogonal systems determine the same involution range of ideal points on the line at infinity, and consequently the double rays of every orthogonal system pass through the same pair of imaginary points on this line.

For reasons that will be given in Chap. XIII these points are called *the circular points at infinity*, or shortly, *the circular points*, as it will be shewn there that all circles pass through them. The lines joining a real origin to these points are called isotropic lines, and their equations are $y = \pm ix$, where i is the imaginary quantity $\sqrt{-1}$, and as these points are imaginary and lie on these lines we will denote them by the letters i, i'. At present we wish merely to direct the student's attention to the fact that they may be considered as perfectly definite, though imaginary; their connection with circles and conics will be discussed later in Chap. XIII and more fully in Chap. XIX. Here we are not attempting to give any rigid proofs of their uniqueness or their properties, but we thought it would be interesting to the student to have his attention drawn to this pair of remarkable points.

114. *If V is the vertex of an orthogonal pencil, and Va, Va' a pair of conjugate rays, the pencil $V(aa'ii')$ is harmonic, and conversely, if we are given that the pencil $V(aa'ii')$ is harmonic, the angle aVa' is a right angle.* (By Art. 113.)

The latter part of this property may be stated: *An involution pencil having the isotropic lines for double rays is orthogonal.*

EXAMPLES.

1. If two ranges are homographic, and any point P on their cross-axis is joined to the points of division on the ranges, these rays will form two pencils in involution.

[For if the ranges intersect in the point A, then in both pencils the cross-axis will have PA for its corresponding ray.]

2. If e, f are the double points of the involution whose characteristic is a, a'; b, b', shew that $(ab', a'b, ef)$ form a system in involution, also $(ab, a'b', ef)$ form a third system in involution, whose double points, if real, are conjugate points of the first system.

Of these three systems formed by taking all possible pairs of the characteristic points a, a', b, b', two have real double points, and one has them imaginary, and the double points of each system are conjugate points of the other two.

3. Given two homographic pencils, centres O, O', shew that any transversal through the cross-centre T will be cut by the pencils in two ranges in involution.

In Fig. 27, p. 43, let any transversal through T cut OO' in P. Then in both ranges P has T for its corresponding point.

4. Three fixed points A, B, C are given on a straight line, on which two other points D, E are taken so that

$$(ABCD)=\lambda, \quad \text{and } (ABCE)=\frac{a\lambda+b}{a'\lambda+b'},$$

where a, b, a', b' are constants, and λ a variable parameter.

Shew that D, E will be conjugate points of an involution if $a+b'=0$.

[Let $$AC=p, \quad BC=q, \quad \therefore AB=p-q,$$

$$AD=x, \quad \therefore BD=x-p+q; \quad AE=y, \quad \therefore BE=y-p+q,$$

$$\therefore \lambda=(ABCD)=\frac{AC}{AD}:\frac{BC}{BD}=\frac{p}{x}:\frac{q}{x-p+q},$$

$$\frac{a\lambda+b}{a'\lambda+b'}=(ABCE)=\frac{AC}{AE}:\frac{BC}{BE}=\frac{p}{y}:\frac{q}{y-p+q}.$$

Substituting for λ we obtain a relation between x and y of the form

$$Pxy+Q(x+y)+R=0.]$$

CHAPTER X

INVOLUTION AND HARMONIC SECTION. HARMONIC PROPERTIES OF A QUADRANGLE AND QUADRILATERAL. POLE AND POLAR

Relation between involution and harmonic section.

115. ONE of the most important properties connected with a system in involution is that of Art. 99, which tells us that $(aa'ef)$ is a harmonic range, so that if aa', bb', cc' are pairs of conjugate points forming a system in involution, of which e, f are the double points, we may say that the axis of the involution is harmonically divided at the points aa', bb', cc', ... for the points e, f, and it follows by Art. 108 *that when two segments are harmonic conjugates for a third segment, one of them is entirely within or entirely without the other when the third segment is real; but if the third is imaginary, the other two will overlap.*

Also, given three pairs of points aa', bb', cc' in involution, if e, f are harmonic conjugates for a, a' and b, b', then e, f are the double points of the involution, and are therefore harmonic conjugates for c, c'.

116. By Art. 110 we see that if aa' is any given pair, and mm' a variable pair of conjugate points,

$$am \,.\, am' - aO\,(am + am') + aO \,.\, aa' = 0,$$

and when m and m' coincide, the double points are given by

$$am^2 - 2aO \, . \, am + aO \, . \, aa' = 0 \quad \ldots\ldots\ldots\ldots\ldots(1).$$

Now whether the system is overlapping or non-overlapping, the point O is always real, for its position can always be found by Art. 102. Therefore the product $aO \, . \, aa'$ is also always real, and consequently the product of the roots of the equation (1), *i.e.* the product $ae \, . \, af$, is always real.

Again, suppose we have given two real points e, f on a line, if we take any other point a on the line we can always find a' its harmonic conjugate for e and f, *i.e.* we can find an infinite number of pairs of conjugate points aa', bb', cc', ... such that each pair taken with ef forms a harmonic range; and we are merely expressing the same geometrical fact in a different way when we say that aa', bb', ... form a system in involution in which e, f are the double points. Also, in the case where e, f are a pair of imaginary points on the line, and a any real point on it, if we know the position of O, and the value of the product $ae \, . \, af$, *i.e.* the value of $aO \, . \, aa'$, we can always find a' the harmonic conjugate of a for e, f. So that we can either commence with a system in involution, and proceed to find the double points, which will be imaginary or real according as the system is overlapping, or non-overlapping; or we can begin with the double points and from them construct the involution system which will be overlapping or non-overlapping according as the double points are imaginary or real.

In Fig. 52, if we suppose the range carrying the accented letters to rotate about O through two right angles, so as to bring a', b', ... into the positions a'', b'', ..., we shall have

$$\begin{aligned} Oa \, . \, Oa'' = Ob \, . \, Ob'' = \ldots &= - Oe^2 = - Of^2 \\ &= OE^2 = OF^2. \end{aligned}$$

The points a, a''; b, b''; ... will be pairs of harmonic conjugates for the imaginary points E, F, whose mid-point O is real.

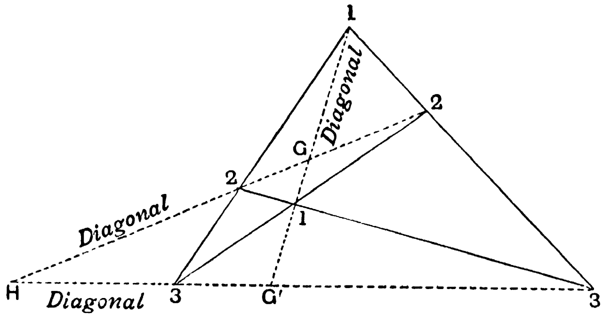

Fig. 53.

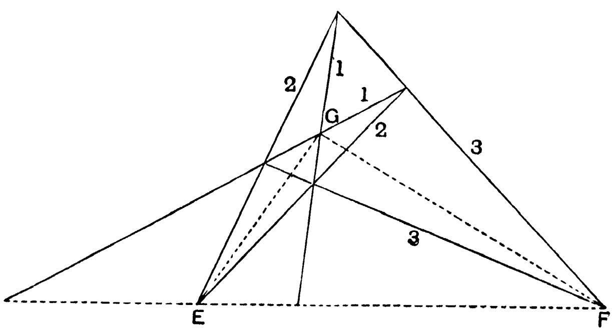

Fig. 54.

117. As we shall frequently in the following pages have to deal with quadrilaterals and quadrangles, we will remind the student that a quadrilateral is a collection of four lines, no three of which pass through the same point. If we call the intersection of any two of these lines a vertex, there are three pairs of opposite vertices, the lines joining which are called diagonals, or diagonal lines, and form the sides of a diagonal triangle, and the whole figure is called a complete quadrilateral.

Quadrilateral, Quadrangle.

A quadrangle is a collection of four points, no three of which lie on the same straight line. If we call the line joining any two of these points a side, there are three pairs of opposite sides, the intersections of which are called diagonal points and are the vertices of a diagonal triangle, and the whole figure is called a complete quadrangle.

Fig. 53 shews us the complete quadrilateral with its 4 lines, 6 vertices, and 3 diagonal lines joining pairs of opposite vertices forming the diagonal triangle $GG'H$.

In Fig. 54 we have the complete quadrangle with its 4 points, 6 lines and 3 diagonal points forming the diagonal triangle GEF.

In Fig. 55, if $ABCD$ is taken as a quadrangle, its diagonal triangle is GEF, but if we consider $ABCD$ as a quadrilateral its diagonal triangle is $GG't$. In the same figure it will be noticed that in the quadrilateral and the quadrangle the diagonal triangles have the same vertex G, and their bases are in the same straight line. It is arbitrary which quadrilateral (of three) we take in conjunction with a given quadrangle, and this accounts for the want of symmetry in the relations between the two.

Harmonic properties of a quadrangle and quadrilateral.

118. *In a complete figure such as Fig. 55:*

(1) *In the quadrangle $ABCD$ the three pairs of opposite sides cut harmonically the three sides of its diagonal triangle GEF, i.e. the ranges $(EsGr)$, $(FqGp)$, $(FG'Et)$ are harmonic.*

(2) *In the quadrilateral $ABCD$ the three pairs of opposite vertices divide harmonically the three sides of its diagonal triangle $GG't$, i.e. the ranges $(CGAt)$, $(DGBG')$, $(FG'Et)$ are harmonic.*

The extremities of the bases EF, tG' of the two diagonal triangles form a harmonic range $(FG'Et)$.

[It will be shewn also that every line in the figure is divided harmonically. The proofs of the foregoing properties are given without reference to any special order. Owing to the particular quadrilateral that has been taken in conjunction with the quadrangle the third range in (1) is identical with the third range in (2), while the others are distinct.]

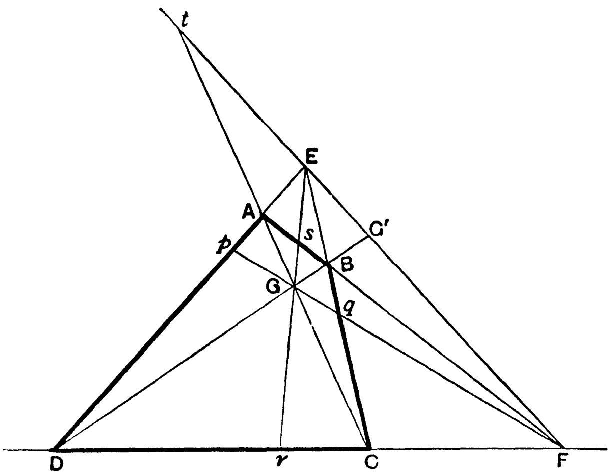

Fig. 55.

The ranges $(tAGC)$, $(tEG'F)$ are in perspective, centre D; therefore by Art. 21 their cross-ratios are equal. Also the ranges $(tAGC)$, $(tFG'E)$ are in perspective, centre B. Therefore their cross-ratios are equal.

$$\therefore\ (tEG'F) = (tFG'E),$$

i.e. the range $(tEG'F)$ being unchanged in value when the pair of points E, F are interchanged separately, is harmonic by Art. 28, and, consequently, so also is $(tAGC)$, and $F(tAGC)$ is a harmonic pencil, which therefore cuts every transversal in a harmonic range by Art. 21, and hence the following ranges are all harmonic, viz. $(DpAE)$, $(rGsE)$, $(CqBE)$ and $(DGBG')$.

Similarly, since $E(DGBG')$ is a harmonic pencil, the following are all harmonic ranges, viz. $(AsBF)$, $(pGqF)$ and $(DrCF)$.

We have proved incidentally the property

When two equicross ranges of four points intersect in one of the pairs of corresponding points, the condition that they should be harmonic is that they should be in perspective at two different centres.

Of course, if two ranges are in perspective, and if we know that one of them is harmonic, the other will also be harmonic by Art. 21.

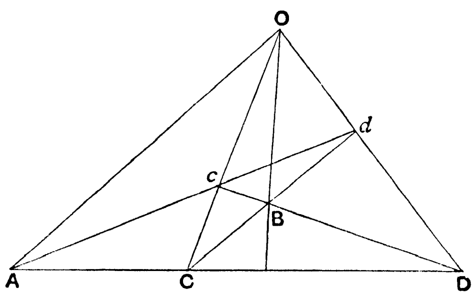

Fig. 56.

The results of this article also give us a method of drawing a fourth ray of a harmonic pencil when three of them are given.

(1) Suppose we have given the pairs OC, OD and OA.

Through A draw two transversals ACD, Acd.

Join Cd, cD, intersecting in B. Then OB is the conjugate of OA.

(2) Suppose we have given OC, OD and OB.

On OB take any point B, and through it draw two transversals cBD, CBd. Join cd, CD, intersecting in A. Then OA is the conjugate of OB.

119. *Any transversal drawn across a quadrangle cuts its three pairs of opposite sides in six points which form a system in involution**.

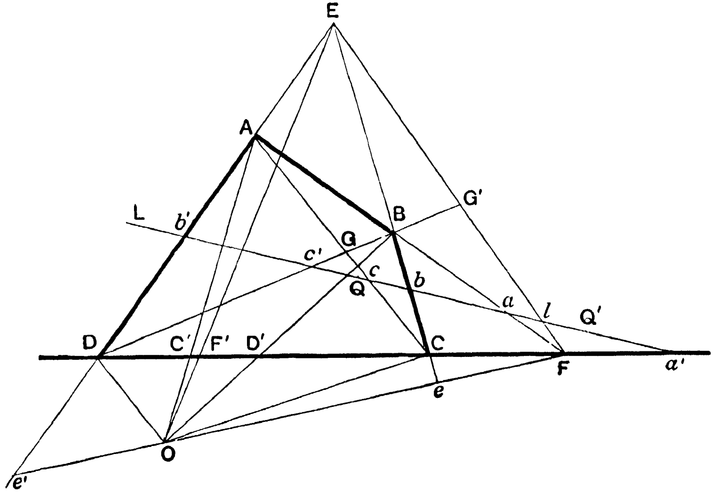

Fig. 57.

Let L be the transversal, and let it cut the equal pencils $A\,(BGc'D)$ and $C\,(BGc'D)$ in the ranges $acc'b'$ and $bcc'a'$.

Then $(acc'b') = (bcc'a') = (a'c'cb)$ by Art. 3.

Therefore by rule of Art. 105, aa', bb', cc' are six points in involution.

120. *The six lines drawn from any point to the three pairs of opposite vertices of a quadrilateral form a pencil in involution.*

In Fig. 57 let O be the given point, and take the pencils joining it to the ranges $(AEe'D)$, $(BEeC)$, which are in perspective.

We have $O\,(AEFD) = O\,(BEFC) = O\,(CFEB)$.

Therefore by the rule of Art. 105, the pencil $O\,(AC, BD, EF)$ is in involution.

* Pappus, Bk VII, Prop. 130. See also Appendix I, p. 124.

Pole and Polar.

121. Another relation which is intimately connected with harmonic section is that of pole and polar, which we shall find of the greatest importance when we come to treat of conics. For the present we give the following:

Pole, Polar.

DEF. If we have an angle BAC and any point O in its plane, and if we draw AP the harmonic conjugate of AO for the lines AB, AC, then AP is called the *polar* of O, and O is called the *pole* of AP for the lines AB, AC or for the angle BAC. See Fig. 58.

122. *The polars of a point for two angles of a triangle intersect on the line which joins the point to the third vertex of the triangle.*

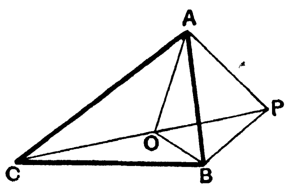

Fig. 58.

Let O be the given point, AP the polar of O for the angle A, and BP the polar of O for B. Then shall CO pass through P.

For the pencils $A\,(COBP)$ and $B\,(COAP)$ are harmonic, and consequently equicross, and they have the common ray AB. Therefore, by Art. 25 their intersections C, O, P are collinear.

123. *Let AO, BO, CO be any three concurrent lines through the vertices of a triangle, meeting the opposite sides in α, β, γ respectively, and let $\beta\gamma$ meet BC in α'. Then $A\alpha'$ is the polar of O for the angle BAC.*

For the ranges $(BaCa')$ and $(\gamma\delta\beta a')$ are equicross, being in perspective, centre A. And the ranges $(BaCa')$ and $(\beta\delta\gamma a')$ are equicross, being in perspective, centre O.

Therefore $(\gamma\delta\beta a') = (\beta\delta\gamma a')$.

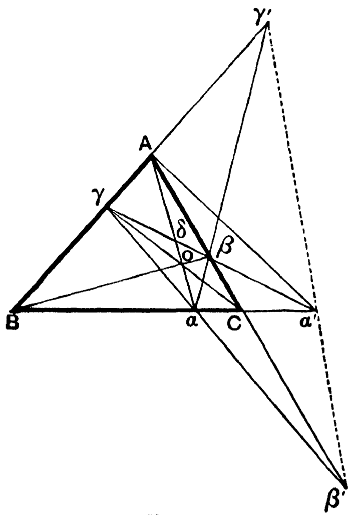

Fig. 59.

Therefore by Art. 28 the range $(\gamma\delta\beta a')$ is harmonic, and therefore so also is the range $(BaCa')$ *i.e.* Aa' is the polar of O for AB, AC.

Similarly, if γa, $a\beta$ meet CA, AB respectively in β', γ', $B\beta'$ is the polar of O for the angle ABC, and $C\gamma'$ is the polar of O for the angle ACB.

124. *The polars of a given point for the three angles of a triangle meet the opposite sides in three points which are collinear.*

Let O be the given point in Fig. 59, and produce AO, BO, CO to meet the opposite sides in a, β, γ, and let the sides of the triangle $a\beta\gamma$ be produced to meet BC, CA, AB in a', β', γ' respectively. Then Aa', $B\beta'$, $C\gamma'$ are the polars of O for the angles A, B, C.

Now the triangles $\alpha\beta\gamma$, ABC being co-polar, are co-axial, by Art. 26. Therefore α', β', γ' are collinear.

Conversely, if a transversal meets the sides BC, CA, AB of a triangle in α', β', γ', and on these sides we take α, β, γ the harmonic conjugates of α', β', γ' respectively, the lines $A\alpha$, $B\beta$, $C\gamma$ are concurrent.

Cor. The sides BC, CA, AB are harmonically divided at α, α'; β, β'; γ, γ'; and these six points lie by threes on the four straight lines $\alpha'\beta'\gamma'$, $\alpha'\beta\gamma$, $\alpha\beta'\gamma$, $\alpha\beta\gamma'$.

125. *In Fig. 57 if Q is any point in the plane of the quadrangle $ABCD$, the polars of Q for the angles DEC, AFD, DGC are concurrent.*

Let EQ' be the polar of Q for the angle AEC, and FQ' its polar for AFC. Join QQ' cutting the three pairs of opposite sides of the quadrangle in the points aa', bb', cc', which by Art. 119 form a system in involution.

Then since by Art. 31 Q, Q' are harmonic conjugates for a, a', and also for b, b', they are the double points of the system in involution to which a, a' and b, b' belong. But c, c' belong to the same system. Therefore by Art. 115 Q, Q' are harmonic conjugates for c, c', *i.e.* the polar of Q for the angle CGD passes through the point Q'.

EXAMPLES.

1. The lines joining the vertices of a triangle to the mid-points of the opposite sides are concurrent.

[In the converse of Art. 124 suppose the transversal to be at infinity.]

2. In Fig. 57 let the transversal L meet EF in l, and on EF take λ the harmonic conjugate of l for the points E, F.

Similarly on AC take γ the harmonic conjugate of c for A, C, and on BD take γ' the harmonic conjugate of c' for B, D. Then will λ, γ, γ' be collinear.

[Let $\lambda\gamma$ meet BD in γ'', and L in P. Then by Art. 120, $P(EF, AC, BD)$ is an involution pencil; and by construction $P(EF, l\lambda)$ and $P(AC, c\gamma)$, *i.e.* $P(AC, l\lambda)$, are harmonic. $\therefore$ $P(BD, l\lambda)$, *i.e.* $P(BD, c'\gamma'')$, is harmonic. Therefore γ'' coincides with γ'.]

3. The mid-points of the three diagonals of a complete quadrilateral are collinear.

[In Fig. 57 let $ABCD$ be the quadrilateral, AC, BD, EF its three diagonals. In Ex. 2 let the transversal L be the line at infinity. Then l, c, c' are at infinity, and λ, γ, γ', the mid-points of EF, AC, BD, are collinear.]

4. Shew that the three points in which the external bisectors of the angles of a triangle meet the sides produced are collinear.

5. The corresponding sides bc, $b'c'$, etc. of two triangles abc, $a'b'c'$ in plane perspective intersect in α, β, γ respectively, and aa', bb', cc' respectively intersect the line $\alpha\beta\gamma$ in α', β', γ'.

Prove that the range $(\alpha\alpha', \beta\beta', \gamma\gamma')$ forms a system in involution. Use Fig. 12, p. 20.

6. ABC is a triangle, and O any point in its plane. Prove that the external bisectors of the angles BOA, AOC, COB intersect the sides AB, AC, BC respectively in three points which lie on a straight line.

7. The lines OA', OB', OC' bisect the internal angles formed by the lines joining any point O to the angular points of the triangle ABC, meeting BC in A', CA in B' and AB in C'. Also A'', B'', C'' are harmonic conjugates of A', B', C' for B and C, C and A, A and B. Prove that A'', B'', C'' are collinear.

8. If a circle is described about a triangle, the points where the tangents at its vertices meet the opposite sides are collinear.

9. Prove that if ABC, DEF are two triangles, and if S is a point such that SD, SE, SF cut the sides BC, CA, AB respectively in three collinear points, then SA, SB, SC will cut the sides EF, FD, DE respectively in three points which are collinear.

10. If through O, the intersection of the diagonals of a quadrilateral $ABCD$, a line OH is drawn parallel to the side AB meeting CD in G and the third diagonal in H, prove that OH is bisected at G.

APPENDIX I

PAPPUS' ACCOUNT OF THE PORISMS OF EUCLID, AND HIS LEMMAS (I—XIX) ON THEM

I PROPOSE to give in this Appendix a short account of Euclid's Treatise on Porisms, the loss of which is probably more to be regretted than that of any of the other treatises on geometrical analysis which were extant in the time of Pappus (400 A.D.) but which have since disappeared. The reason why reference should be made to it in the present work will be obvious to anyone who glances in the most cursory manner over the Lemmas which Pappus gave as explanatory propositions to it. From them one can hardly fail to draw the conclusion that the master mind which conceived the Porisms was quite familiar with the fundamental principles of homography, I mean with harmonic section, the harmonic properties of a quadrilateral, homographic ranges and pencils, and involution. Unfortunately the ancient geometers suffered from three hindrances, viz. (1) the non-recognition of continuity, (2) the non-recognition of sense in the direction of lines and description of angles, and (3) the absence of a suitable notation. It is due in a great measure to the removal of these fetters that homography has been enabled to make the strides which it has done in the last 100 years.

As our knowledge of the Treatise on Porisms is almost confined to the description given by Pappus in the preface to the Seventh Book of the Mathematical Collections, and as this has not up to the present time, as far as I am aware, been given in

English, I have thought it advisable to give a translation of Pappus' account in the hope that it may not only serve the purpose of allowing the student to obtain his information on the subject from the fountain head, but may also perchance be the means of enabling some one versed in Oriental languages to recognise a copy, perhaps in an Eastern dress, in one of our public or private libraries, where it is quite possible that one may be lying *involutus pulvere magis quam tenebris suis**.

[*Translation.*]

"The Three Books of Porisms.

"After the books on contacts (by Apollonius) come the Porisms of Euclid in three books, a most ingenious collection for solving more difficult problems of which the nature of the subject provides an unlimited number. No addition has been made to them as Euclid first wrote them, except that certain stupid persons before our time have given alternative versions in the case of a few of them, for each porism has a definite number of ways in which it can be stated, as we have pointed out, and Euclid has given only one in each case, and that the most obvious one. They are in principle subtle and natural, and indispensable and quite general, and afford much pleasure to those who are able to understand and investigate them.

Porisms of all classes are neither theorems nor problems, but they occupy a position intermediate between the two, so that their enunciations can be stated either as theorems or problems, and consequently some geometers think that they are really theorems, and others that they are problems, being guided solely by the form of the enunciation. But it is clear from the definitions that the old geometers understood better the difference between the three classes. For they said that a theorem is that

* Cf. p. 75.

in which something is proposed for demonstration, a problem is that in which something is proposed for construction, and a porism is that in which something is proposed for [discussion or] investigation. This definition has been changed by later writers, who were not able to fully investigate them, but as is usual in the Elements [of Euclid] they only gave a demonstration of the quæsitum, without also giving a discussion of it. And although they are shewn to be mistaken by the definition given above, and by what is known of the subject, they gave a definition somewhat as follows: A porism is that which in the hypothesis is less complete than a local theorem. And loci are instances of this class of porisms, and they abound in analysis. But this class of questions, because it has a wider range than other classes, has been separated from porisms, and a collection of them has been made, and treatises written on them and handed down to us. And of these loci some are plane, some are solid, some are linear, and some depend on mean proportionals.

Now it sometimes happens that porisms have enunciations which are contracted owing to the abbreviated form of expression, and in them much is generally supposed to be supplied (by the reader), so that many geometers only partially understand the matter, and do not comprehend the more important part which is implied in them. And in the case of porisms it is not possible to include many in one proposition because Euclid himself has not given many out of each class; but for the sake of example out of a great number he has given a few belonging to the same class at the beginning of his first book, all of them, about 10 in number, belonging to that somewhat numerous class of loci; wherefore finding it possible to include these in one statement, we have given it as follows*:

In Fig. 22, p. 36, given four straight lines BC, CA, AB, DE

* For the Greek text of this Porism which is given by Pappus without either figures or letters, see p. 121.

intersecting by pairs in the points A, B, C, P, Q, R, if three of the points P, Q, R lying on one of them DE (or two of them in the case of parallelism), [*i.e.* when BC is parallel to DE, in which case P is at infinity], are fixed, and of the other three points two, viz. B and C, move along the fixed straight lines OD, OE, the last point A will also move along a fixed straight line.

This enunciation refers to only four straight lines, of which not more than two pass through the same point, and does not mention the fact that a similar proposition holds for any proposed number, which may be stated as follows: If any number of straight lines cut one another, of which not more than two pass through the same point, and it is also given that all the points of intersection lying on one of them are fixed, and of those which lie on the others each moves along a given straight line; or more generally as follows: If any number of straight lines cut one another, of which not more than two pass through the same point, and it is also given that all the points of intersection lying on one of them are fixed, and if the number of the rest is a triangular number whose side is the same as the number of the fixed collinear points, and no three of them (*i.e.* of the points of intersection of the moving lines with the given lines) are at the vertices of a triangle, then each of the remaining points describes a straight line.

Now it is not likely that the writer of the Elements was not aware of this, but he was merely stating the first principles, and in the case of each of the porisms he seems to have put forth only the first principles and germs of many important properties, and of these the classes are to be distinguished not by the differences of their hypotheses, but by those of their conclusions and quæsita*. For all hypotheses, being very special in character,

* Simson (*Rel. Op.* p. 349) explains this to mean that there are many porisms which have different hypotheses, but in all of which the conclusion is 'that a certain point lies on a fixed straight line' or 'that a certain straight line passes through a fixed point,' &c.

differ from each other, and each conclusion and quæsitum, although one and the same, are to be considered separately in the many different hypotheses. Now in the first book the following classes are to be formed by the quæsita in the propositions. There is a diagram referring to this at the beginning of the seventh book*.

[Classes of Porisms.]

I. If from two given points (two) straight lines are drawn intersecting on a given straight line, and if one of them cuts off from a given straight line (a segment measured) from a given point in it, the other will also cut off from another straight line (a segment measured from a given point in it) having a given ratio (to the former segment)†.

II. A certain point lies on a given straight line.

III. The ratio of a certain line to a certain other line is given.

IV. The ratio of a certain line to a segment (is given).

V. A certain line is given in position.

VI. A certain line passes through a given point.

VII. The ratio of a certain line to a segment between a certain point and another given point (is given).

VIII. The ratio of a certain line to a segment drawn from a certain point (is given).

IX. The ratio of a certain rectangle to that contained by a given line and a certain other line (is given).

X. Of a certain rectangle one part is given, and the remainder has a given ratio to a segment of a line.

XI. A certain rectangle, or a certain rectangle together with another given rectangle is (given), and the former has a (given) ratio to a segment of a line.

* This diagram is unfortunately missing.

† For proof see Simson, *de Porismatibus*, p. 400, Prop. 23, and Chasles, *Porismes d'Euclide*, p. 114, Prop. 11.

XII. A certain line together with another line with which a certain other line is in a given ratio is itself in a (given) ratio to a segment drawn from a certain point to a given point.

XIII. (A triangle whose vertex is) at a given point, and (whose base is) a certain straight line is equal to (a triangle whose vertex is) at a given point, and (whose base is a segment drawn) from a certain point to a given point.

XIV. The ratio of the sum of a certain pair of lines to a segment drawn from a certain point to a given point (is given).

XV. A certain line cuts off from two given lines (segments) which have a (given) rectangle.

In the second book the hypotheses are different, but the greater number of the quæsita are the same as those in the first book with these additional ones.

XVI. A certain rectangle or a certain rectangle together with a given rectangle has a (given) ratio to a segment of a line.

XVII. The ratio of the (rectangle contained) by certain lines to a segment of a line (is given).

XVIII. The ratio of the rectangle, one of whose sides is the sum of a certain pair of lines and the other side the sum of a certain other pair of lines, to a segment of a line (is given).

XIX. The rectangle, one of whose sides is a certain line and the other side the sum of a certain line and of another one to which a certain line bears a given ratio, and the rectangle whose sides are a certain line and another line to which a certain line bears a given ratio have their sum in a (given) ratio to a segment of a line.

XX. The ratio of the sum of two rectangles to a segment drawn from a certain point to a given point (is given).

XXI. The rectangle contained by a certain pair of lines is given.

In the third book the majority of the hypotheses relate to semicircles, and a few to the circle and segments. Most of the quæsita are similar to those given above, with these additional ones.

XXII. The ratio of the rectangle contained by a certain pair of lines to that contained by a certain other pair of lines (is given).

XXIII. The ratio of the square on a certain line to a segment (is given).

XXIV. The rectangle contained by a certain pair of lines (is equal) to the rectangle contained by a given line and a line drawn from a certain point to a given point.

XXV. The square on a certain line (is equal) to the rectangle contained by a given line and an abscissa of a line between a given point on it and the foot of a perpendicular.

XXVI. The sum of a certain line and of a line to which a certain other line has a given ratio, has a (given) ratio to a segment.

XXVII. There exists a certain given point from which straight lines drawn to certain (circles) will enclose a triangle of given species.

XXVIII. There exists a certain given point from which straight lines drawn to a certain (circle) cut off equal arcs.

XXIX. A certain straight line is either parallel to, or makes a given angle with a straight line drawn to a given point.

The three books of porisms have 38 lemmas, and they contain 171 theorems."

Poncelet in the introduction to his *Traité de propriétés projectives des figures* (1822) suggested that the porisms were projective properties deduced by Euclid from considerations of perspective, but Chasles has made it pretty clear that Euclid's treatise was concerned with the principles of cross-ratios, Book I dealing with homographic divisions on two lines, Book II with

co-axial ranges, and Book III with the anharmonic properties of the circle.

In order that the student may understand the nature of the difficulty which presented itself to the geometers of the 17th, 18th, and 19th centuries, several of whom attacked the question, we give the Greek text of the only Porism enunciated by Pappus.

ἐὰν ὑπτίου ἢ παρυπτίου τρία τὰ ἐπὶ μιᾶς σημεῖα [ἢ παραλλήλου ἕτερα τὰ δύο] δεδομένα ᾖ, τὰ δὲ λοιπὰ πλὴν ἑνὸς ἅπτηται θέσει δεδομένης εὐθείας, καὶ τοῦθ' ἅψεται θέσει δεδομένης εὐθείας.

A paraphrase of the above general proposition is given on p. 116, with letters and a reference to a figure. Simson explains the term *ὕπτιον* to mean a quadrilateral in which two adjacent sides tend to meet in a direction opposite to that in which the others tend to meet, whilst *παρυπτίον* is a quadrilateral in which two adjacent sides tend to meet in the same direction as the others, *e.g.* in Fig. 22, p. 36, $ABOC$ is an example of *ὕπτιον*, since BO, CO tend to meet in the opposite direction to BA, CA, and $PQAC$, $PBAR$ are examples of *παρυπτίον*, for PQ, AQ tend to the same direction as PC, AC, and similarly PB, AB tend to the same direction as PR, AR.

The whole subject was an enigma to Fermat (1601—1665), and even Halley (1706) confesses "Porismatum descriptio nec mihi intellecta, nec lectori profutura. Quid sibi velit Pappus haud mihi datum est conjicere." It was not until 1723 that the key to the mystery was found by R. Simson (1687—1768), professor of mathematics at Glasgow, and his account of his discovery will always be read with interest. "I often tried," he said, "but always in vain, to understand and restore the only porism which survives out of all that were in the three books, and as my meditations on it took up too much of my time I determined that I would never touch the subject again, especially as Halley had given up all hope of understanding it. Consequently, whenever it occurred to me, I always refused to dwell upon it. However, sometime afterwards it presented itself to my mind when I was

off my guard, and had in fact forgotten all about it, and it held possession of my thoughts until at length a glimmer of light was thrown upon it which gave me hopes of discovering Pappus' general proposition, and this, after much thought, I was at length enabled to restore."

In his *Opera quædam reliqua* published in 1776, eight years after his death, we find a restoration of Euclid's treatise containing 93 propositions*. This roused a fresh interest in the subject, and in 1860 Chasles published his restoration "Conformément au sentiment de R. Simson sur la forme des énoncés de ces propositions." For further information we must refer the student to Chasles' work *Les trois livres de Porismes d'Euclide*, 1860, 324 pp. in which he gives 219 Porisms, with a complete historical account of Euclid's treatise, and demonstrates its homographical character.

The following are Props. 127—144 of the 7th book of Pappus. The enunciations are the literal translations of the Greek text, but I have substituted proofs which are intended to shew the connection of the propositions with the theory of cross-ratio.

LEMMAS ON EUCLID'S PORISMS.

On the 1st Porism of Bk I.

I. *In Fig.* 60 *let* $ac : cb' = ab : bc'$, *and let* CD *be joined. I say that the straight lines* ac', CD *are parallel.*

* On the outside cover of an Appendix (1847) to Potts' larger edition of Euclid there was a notice that it was proposed to publish by subscription a translation of Simson's *Restoration of the Porisms*. The translation was to be preceded by a discussion of their peculiar character, together with a full development of the algebraical method of investigating them.

If a number of subscribers had been obtained sufficient to defray expenses, it was intended to print the work at the University Press in octavo, and to issue it at a price not exceeding ten shillings.

[Consider ab as a transversal of the quadrilateral $ABCD$. The given relation is equivalent to

$$ac : ab' = ab : ac', \text{ or } ac \,.\, ac' = ab \,.\, ab'.$$

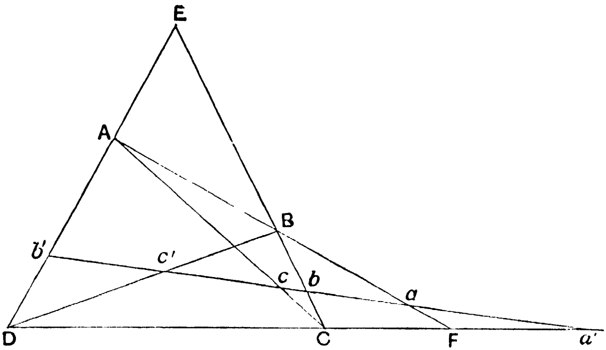

Fig. 60 (I, II, IV).

Therefore a is the centre of the involution system of points in which the transversal meets the sides and diagonals of the quadrilateral.

Therefore a', the conjugate of a, is at infinity, and the transversal is parallel to CD. See Arts. 98, 119.]

On the 2nd Porism.

II. *In the same figure let* ab *be parallel to* CD, *and let* $bc' : ba = cb' : ca$. *I say that the points* A, B, a *are collinear.*

[The relation is equivalent to

$$ac' : ab = ab' : ac, \text{ or } ab \,.\, ab' = ac \,.\, ac';$$

therefore the conjugate of the point at infinity is a; *i.e.* the transversal meets the side AB in the point a. Art. 119.]

III. *Two straight lines* ad, ad' *are drawn cutting the three straight lines* Ob, Oc, Od. *I say that*

$$ab' \,.\, d'c' : ad' \,.\, c'b' = ab \,.\, dc : ad \,.\, bc.$$

[*i.e.* $(ac'b'd') = (acbd)$. Art. 21.]

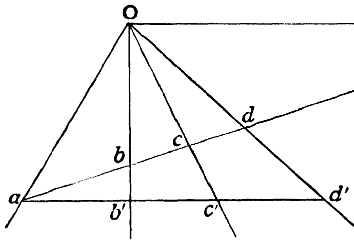

Fig. 61 (III, X, XI, XIV, XVI).

IV. *In Fig.* 60 *let* $aa' \,.\, bc : ac \,.\, ba' = aa' \,.\, b'c' : ab' \,.\, c'a'$. *I say that* C, D, a' *are collinear.*

[Otherwise, a transversal meets two diagonals of a quadrilateral in c, c', two opposite sides in b, b' and a third side in a. If a point a' is taken on the transversal such that $(a'cab) = (a'b'ac')$ the transversal will meet the fourth side in the point a'. Art. 119.]

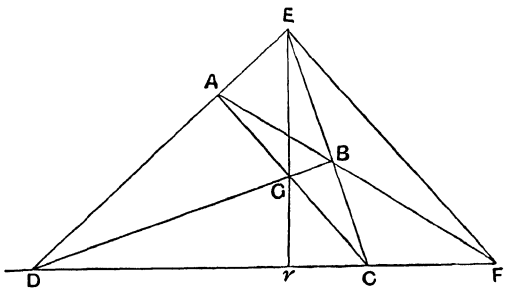

Fig. 62 (V, VI).

V. *In Fig.* 62 *(it is proved elsewhere that)* $DF : FC = Dr : rC$. *Therefore if* $DF : FC = Dr : rC$, *I say that* D, G, B *are collinear.*

VI. *In Fig.* 62 *if* AB *and* DC *are parallel, (it is proved elsewhere that)* $Dr = rC$. *Suppose now that* $Dr = rC$, *I say that* AB *and* DC *are parallel.*

[$DrCF$ is a harmonic range, and if $Dr = rC$, F is at infinity, *i.e.* AB is parallel to DC. Arts. 118, 29.]

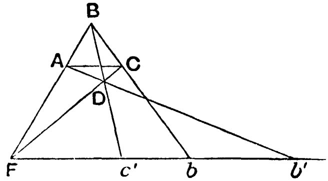

Fig. 63 (VII).

VII. *Let Fc' be a mean proportional between $c'b$ and $c'b'$. I say that AC and Fc' are parallel.*

[$Fc'bb'$ is a transversal of the quadrilateral $ABCD$. F is one of the double points of the involution, and since $c'b \, . \, c'b' = c'F^2$, c' is the centre, and its conjugate c, the intersection of AC and Fc', is at infinity.]

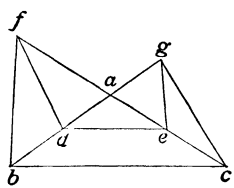

Fig. 64 (VIII).

VIII. *Let abcdefg be a Bomiscus [i.e. a figure like a little altar with unequal sides], and let de be parallel to bc, and eg to bf. I say that fd is parallel to cg.*

[Otherwise, if $bcgedf$ is a hexagon inscribed in the line-pair bdg, cef, and if the opposite sides ge, fb are parallel, and also ed parallel to bc, then shall the remaining pair cg and fd be parallel, *i.e.* the Pascal line is altogether at infinity. See Art. 51.]

IX. *Pth' is a triangle, and in it are drawn Pr, Pk', and AD is drawn parallel to th', and let AF, DF be drawn to a point F on th' such that $tF : Fh' = rF : Fk'$. I say that BC is parallel to th'.*

[By Art. 120 Pt, Pr, Pk', Ph' are four rays of an involution

pencil drawn from P to the four vertices of the quadrilateral $ABCD$.

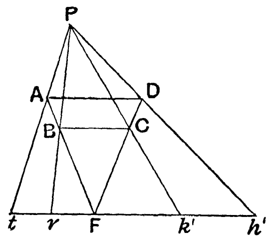

Fig. 65 (IX).

Therefore (t, k') and (r, h') are pairs of conjugate points. Also since $Ft \,.\, Fk' = Fr \,.\, Fh'$, F is the centre of the involution, and therefore its conjugate is at infinity; therefore the sides AD, BC meet at infinity, *i.e.* those sides are parallel.]

X. *In Fig.* 61, *from the point a two straight lines abc, $ab'c'$ are drawn cutting the two lines Obb', Occ', and let d, d' be points on them such that $ca \,.\, bd : cd \,.\, ba = ac' \,.\, d'b' : ab' \,.\, d'c'$. I say that the points O, d, d' are collinear.*

[Since $(cbad) = (c'b'ad')$, $\therefore$ O, d, d' are collinear by Art. 23.]

XI. *In the same Fig.* 61, *let $Ob'c'$ be a triangle, and OD parallel to $b'c'$, and draw Da meeting $b'c'$ in a. I say that*

$$aD \,.\, bc : ab \,.\, cD = c'b' : b'a.$$

[In X, d is at D, and d' at infinity, $\therefore$ $(acbD) = (ac'b'\infty) = \dfrac{ab'}{c'b'}$.]

XII. *Having proved the preceding propositions we will now shew that in Fig.* 66 *if ac', Oe are parallel, and are intersected by the straight lines Oc', $c'd$, da, ae, and the lines Ob', $b'e$ are drawn, I say that the points k, m, b are collinear.*

[This is Pascal's Theorem for the hexagon $Ob'eadc'$ inscribed in the line-pair Ode, $ab'c'$, these lines being parallel. Art. 51.

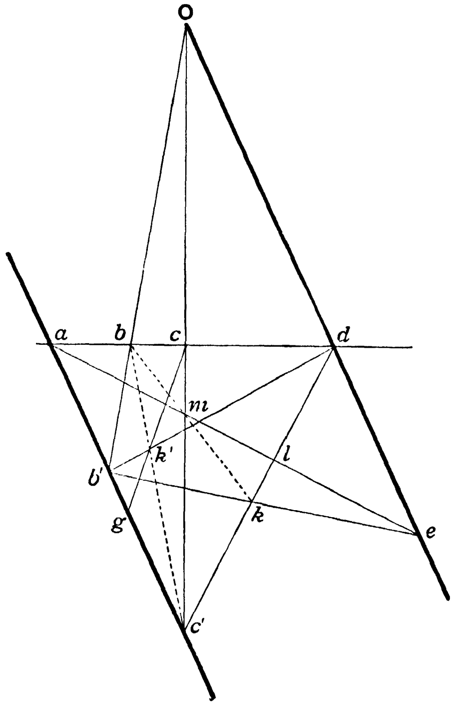

Fig. 66 (XII, XV, XVIII).

Since Ob, Oc, Od are intersected by $abcd$, $ab'c'$, of which the latter is parallel to Od, $\therefore$ by Lemma XI, $(acbd) = \frac{ab'}{c'b'}$.

Similarly, since ed, ea, eb' are intersected by $c'd$, $c'a$, of which the latter is parallel to ed,

$$\therefore\ (c'lkd) = \frac{c'b'}{ab'}. \quad \therefore \text{ by Art. 3, } (acbd) = (lc'kd).$$

And al, cc' meet in m. $\therefore$ by Art. 23, b, m, k are collinear, bmk being the cross-axis. Art. 50.]

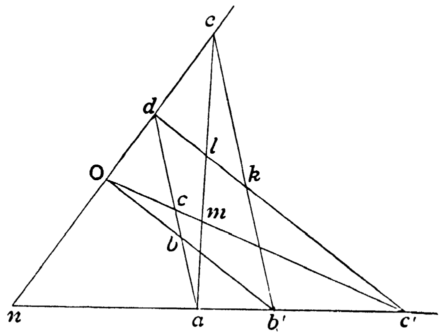

Fig. 67 (XIII, XIX).

XIII. *Now let ac', Oe be not parallel, and let them meet in the point n. I say that in this case also the points k, m, b are collinear.*

[This is Pascal's Theorem for the hexagon $Ob'eadc'$ inscribed in the line-pair Ode, $ab'c'$ in the general case. Art. 51.

Since On, Ob', Oc' are cut by ad, ac',

$$\therefore\ (acbd) = (ac'b'n) \text{ by Lemma III}$$

$$= (lc'kd), \text{ since } en,\ ea,\ eb' \text{ are cut by } c'd,\ c'n.$$

And al, cc' meet in m, $\therefore$ by Art. 23, b, m, k are collinear and bmk is the cross-axis.]

XIV. *In Fig.* 61, *let OD be parallel to ac', and let Ob', aD be drawn, and on bD take a point c such that $c'b' : b'a = aD \,.\, bc : cD \,.\, ab$. I say that the points O, c, c' are collinear.*

[See Lemma XI $(acbD) = (ac'b'\infty)$. Then Art. 23.]

XV. *The previous propositions having been proved, in Fig.* 66 *let ac' be parallel to Oe, and let them be cut by the straight lines $c'd$, da, eb', $b'O$, and draw the lines ae, bk. I say that the points c', m, O are collinear.*

[Otherwise, O, d, e and a, b', c' are two triads of points on two parallel straight lines. Ob', ad meet in b, $c'd$, $b'e$ meet in k, and ae, bk meet in m. Then O, m, c' are collinear. The converse of Lemma XII.]

XVI. *The same as X.*

XVII. *The same as XV, except that Oe, ac' are not parallel.*

[The converse of XIII.]

XVIII. *In Fig.* 66 *$Ob'c'$ is a triangle, and Od is drawn parallel to $b'c'$, and da, cg are drawn so that $ab'^2 : ac' . c'b' = b'g : gc'$. I say that if $b'd$ is drawn, the points b, k', c' are collinear.*

[The given relation is equivalent to $\dfrac{b'g \,.\, ac'}{gc' \,.\, ab'} = \dfrac{ab'}{b'c'} = \dfrac{dc \,.\, ba}{da \,.\, cb}$ by XI, *i.e.* $(agc'b') = (acbd)$. And since cg, $b'd$ meet in k', $\therefore$ b, k', c' are collinear by Art. 23.]

XIX. *In Fig.* 67 *from the point c' are drawn two straight lines $c'd$, $c'n$ cutting the three straight lines, en, ea, eb', and let $c'd : dl = kc' : kl$. I say that $c'n : na = c'b' : b'a$.*

[The harmonic pencil $e\,(c'kld)$ determines a harmonic range on any transversal $c'b'an$ by Art. 31.]

CHAPTER XI

ANHARMONIC PROPERTIES OF POINTS AND TANGENTS OF A CONIC. THE LOCUS *AD TRES ET QUATUOR LINEAS*

126. We will now proceed to apply the principles of homography to conics, and in doing so we shall assume that the student possesses a knowledge of the elementary properties of conics as given in the ordinary text-books, and we will first give two propositions which are, as it were, the foundations on which we shall build. The method we have adopted is due to B. W. Horne, *Quarterly Journal of Mathematics*, Vol. IV, 278, 1861. For other ways of opening the subject see Chasles, *Traité des Sections Coniques* (1865). See also his *Aperçu Historique*, Notes XV, XVI, and his *Traité de Géométrie Supérieure*, Chap. XXV.

127. *A and B are two fixed points on a conic, focus S, and O is any variable point on the curve. OA and OB meet the S-directrix in a, b. Then the angle $aSb = \frac{1}{2} ASB = const.$*

By the focus and directrix definition of a conic,

$$SO : SA = Oa : Aa,$$

$\therefore$ Sa bisects the angle ASO'. $\quad \therefore$ $aSO' = \frac{1}{2} ASO'$.

Similarly, $$SO : SB = Ob : Bb.$$

$\therefore$ Sb bisects the angle BSO'. $\quad \therefore$ $bSO' = \frac{1}{2} BSO'$.

$$\therefore \ aSb = bSO' - aSO' = \tfrac{1}{2}(BSO' - ASO') = \tfrac{1}{2} ASB.$$

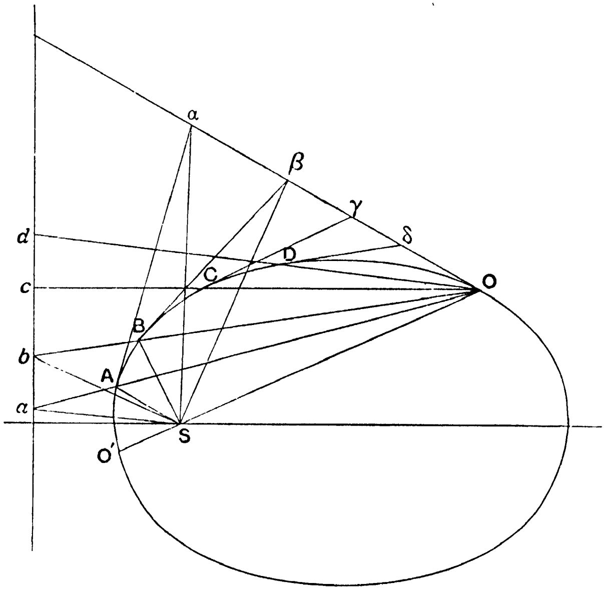

Fig. 68.

128. *If the tangents at A, B meet the tangent at any point O in α, β, the angle $\alpha S\beta = \frac{1}{2} ASB = const.$*

Since tangents to a conic subtend equal angles at a focus, the angle

$$\alpha SO = \alpha SA = \tfrac{1}{2} ASO,$$

and

$$\beta SO = \beta SB = \tfrac{1}{2} BSO,$$

$$\therefore\ \alpha S\beta = \tfrac{1}{2} ASB = aSb.$$

Conic-pencil.

Def. The pencil formed by joining four points on a conic to a fifth point on the curve is called a *conic-pencil**.

* Prof. A. Lodge, 1908.

129. *If A, B, C, D are four fixed points on a conic, and O any variable point on the curve, the conic-pencil $O\,(ABCD)$ has a constant cross-ratio*.*

In Fig. 68 $$O\,(ABCD) = O\,(abcd)$$
$$= S\,(abcd) \text{ by Art. 24,}$$

and by Art. 127 the angles subtended at S by pairs of the points a, b, c, d are half the angles subtended at S by the pairs of corresponding points A, B, C, D, and therefore the pencil $S\,(abcd)$ is constant. Hence the conic-pencil $O\,(ABCD)$ is constant.

As the cross-ratio of the conic-pencil $O\,(ABCD)$ is the same for all points O on the conic, we may speak of it as the cross-ratio of the four points A, B, C, D, and the above property may be stated :

The cross-ratio of four fixed points on a conic is constant, meaning that *the cross-ratio of the conic-pencil formed by joining the four points to any variable point on the curve is constant.*

From the above it follows that if O, O' are two positions of the variable point, the conic-pencil $O\,(ABCD) = O'\,(ABCD)$, and this fact is independent of the position of the points A, B, C, D. Hence we may now suppose O, O' to be two fixed points, and A, B, C, D to be any four positions of a variable point on the conic, and we have the theorem :

If two fixed points on a conic are joined to a variable point on the curve, the pencils so formed are homographic, since any four positions of the variable point give two pencils, centres O, O', having the same cross-ratio.

Hence, considering any six points on a conic,

(a) If we fix four of the points, the other two give us a pair of equicross pencils, and the conic-pencils formed by drawing rays to the four points from any variable point on the curve are equicross.

* Chasles, 1829; Steiner, 1832.

(β) If we fix two of the points and suppose the others variable, the pencils formed by drawing from the two points rays intersecting on the conic are homographic.

130. *If the tangents at four fixed points A, B, C, D on a conic meet the tangent at any variable point O in α, β, γ, δ, the range $(\alpha\beta\gamma\delta)$ has a constant cross-ratio.*

In Fig. 68 by Art. 128 the pencils $S(abcd)$ and $S(\alpha\beta\gamma\delta)$ are superposable. $\therefore S(\alpha\beta\gamma\delta) = S(abcd) = O(abcd) = O(ABCD)$.

Therefore $(\alpha\beta\gamma\delta) = \text{const.}$

As the cross-ratio of the range $(\alpha\beta\gamma\delta)$ is the same for the tangent at any point O of the conic, it may be called the cross-ratio of the four tangents at A, B, C, D, and the above property may be stated:

The cross-ratio of four fixed tangents is constant, meaning that *the cross-ratio of the range in which four fixed tangents cut any variable tangent is constant.*

As in the previous article, we might take two positions TP, TQ of the variable tangent, and let the four tangents at A, B, C, D meet them in α, β, γ, δ and α', β', γ', δ'. Then by the above, $(\alpha\beta\gamma\delta) = (\alpha'\beta'\gamma'\delta')$. This fact is quite independent of the positions of the points A, B, C, D. Hence we may consider TP, TQ as two fixed tangents, and the tangents at A, B, C, D as any four positions of a variable tangent. This gives us the theorem:

If two fixed tangents are cut by a variable tangent the ranges so formed are homographic,

since any four positions of the variable tangent give points on the fixed tangents which have the same cross-ratio. Hence, considering any six tangents to a conic,

(α) If we fix four of the tangents, the other two give us a pair of equicross ranges; and so the ranges formed by the intersections of the four fixed tangents with any variable tangent are equicross.

(β) If we fix two of the tangents, and suppose the others variable, we may remove the restriction to four, and think of any number, and these will determine homographic ranges on the two fixed tangents.

NOTE. The properties in Arts. 129, 130, being projective, could of course be deduced from the corresponding properties of the circle.

131. It follows from the preceding article that:

The cross-ratio of the conic-pencil of any four points on a conic is equal to the cross-ratio of their tangents.

This of course is a very abbreviated statement, and the student should realize that its meaning, expressed in full, is:

The cross-ratio of the conic-pencil formed by joining four fixed points A, B, C, D on a conic to any variable point on the curve is equal to the cross-ratio of the range formed by the points of intersection of the tangents at A, B, C, D with any variable tangent to the conic.

132. The converses of Arts. 129, 130 are very important, and are of two distinct classes. In the first it is assumed that there is given a conic-pencil or a tangent range, and it is proved that equi-pencils or equi-ranges belong to the same conic, and therefore incidentally that only one conic exists in each case. The second class is more general, and does not assume the existence of a conic, but only that of equi-pencils or equi-ranges, and proves that the locus or envelope is a conic, and the theorems of this latter class, which we may term complete converses, really include those of the former, which may be called partial converses.

133. We will now prove the partial converse of Art. 129 (α).

If we have given four fixed points on a conic, and if the pencil formed by joining them to a point P has the same cross-ratio as the conic-pencil formed by the four points, the point P lies on the conic.

Let A, B, C, D be the four fixed points, and suppose P is not on the conic. Let one of the rays PA meet the conic in P'. Then the pencil $P(ABCD)$ = the conic-pencil $P'(ABCD)$, and these have a common ray, and are therefore in perspective,

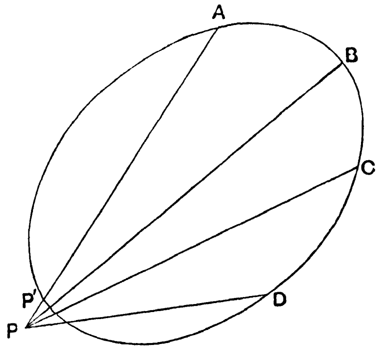

Fig. 69.

i.e. by Art. 25 the points B, C, D are collinear, which is contrary to the supposition that they are on the conic.

COR. If a conic can be drawn through five points, only one conic can be so drawn.

134. *Given four fixed tangents to a conic, if they form on any straight line the same cross-ratio as that formed by them on any fifth tangent, the straight line will touch the conic.*

Partial converse of Art. 130 (a). See Fig. 70.

Let the tangents at the fixed points A, B, C, D intersect the straight line L at a, b, c, d. Then if the line L is not a tangent, from one of the points of intersection, as a, draw a tangent meeting the three other fixed tangents at b', c', d'.

Then the ranges $(ab'c'd')$ and $(abcd)$ are equicross and have a *common point* a, and are therefore by Art. 23 in perspective, *i.e.* the three tangents bb', cc', dd' are concurrent, which is absurd.

COR. If a conic can be drawn to touch five lines, only one conic can be so drawn.

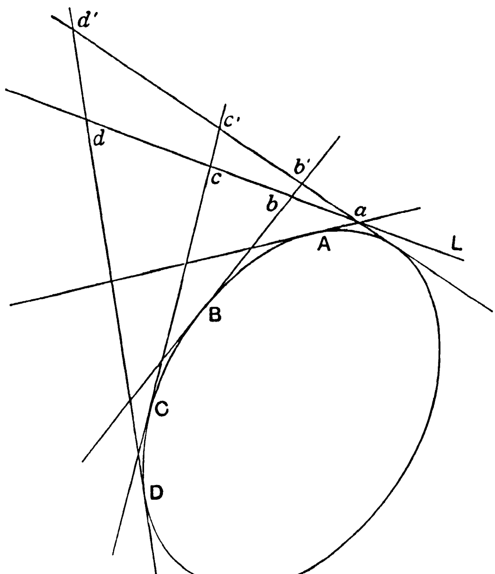

Fig. 70.

Proofs of Arts. 129 (β) and 130 (β) based upon Propositions given by Apollonius.

135. In Apollonius, Bk III, Prop. 54, we find the following property. See Fig. 71.

TI, TJ' are fixed tangents to a conic, P any variable point on the curve. Through I and J' lines are drawn parallel to the tangents, meeting $J'P$, IP in a, a'. Then for all positions of P the rectangle $Ia \,.\, J'a'$ is constant.

Through P draw a line PP' parallel to IJ' meeting the conic in P', the tangents in K, K', and the diameter CT in W. Let

$C\alpha$, $C\beta$, $C\gamma$ be the semi-diameters parallel to IJ', TI, TJ'. To shorten the statement of the proof we shall assume a knowledge of the following properties:

(1) $$KP \,.\, KP' : KI^2 = C\alpha^2 : C\beta^2.$$

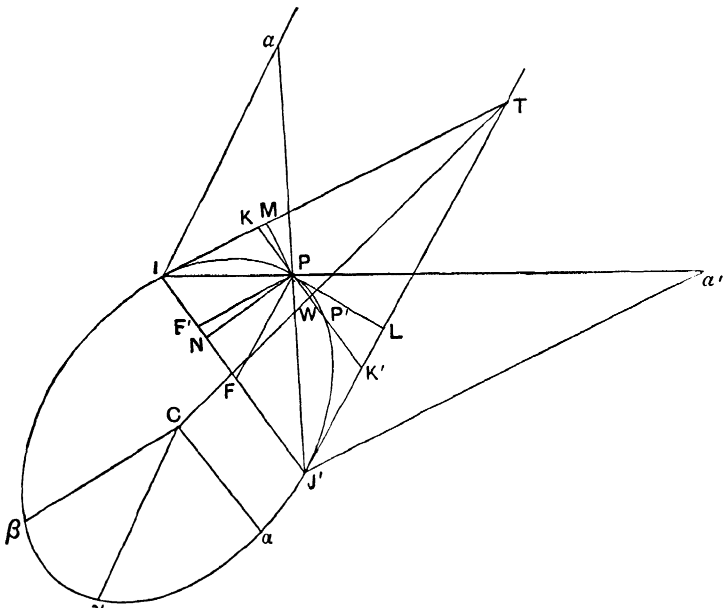

Fig. 71.

(2) W bisects PP', and also KK', and consequently

$$KP = K'P', \quad KP' = K'P,$$

and therefore $$KP \,.\, KP' = K'P' \,.\, K'P = KP \,.\, K'P.$$

(3) From the similar triangles aIJ', $J'K'P$,

$$Ia : IJ' = K'J' : K'P,$$

and from the similar triangles $a'J'I$, IKP,

$$J'a' : IJ' = KI : KP.$$

By (1) $\dfrac{C\beta^2}{C\alpha^2} = \dfrac{KI^2}{KP \, . \, KP'} = \dfrac{KI^2}{KP \, . \, K'P}$ by (2).

Similarly $\dfrac{C\gamma^2}{C\alpha^2} = \dfrac{K'J'^2}{K'P \, . \, K'P'} = \dfrac{K'J'^2}{KP \, . \, K'P}$.

$$\therefore \frac{C\beta \, . \, C\gamma}{C\alpha^2} = \frac{KI}{KP} \cdot \frac{K'J'}{K'P} = \frac{Ia}{IJ'} \cdot \frac{J'a'}{IJ'} \text{ by (3),}$$

$$\therefore Ia \, . \, J'a' = \frac{C\beta \, . \, C\gamma}{C\alpha^2} \, . \, IJ'^2 = \text{const.}$$

Hence by Art. 64 the ranges (a), (a') are homographic, the points I, J' corresponding to the points at infinity, and therefore the pencils $I(a')$, $J'(a)$, *i.e.* the conic-pencils $I(P)$, $J'(P)$, are homographic. Consequently Apollonius' property may be stated :

If I, J', two fixed points on a conic, are joined to any number of points on the curve, the conic-pencils so formed are homographic, which is equivalent to the theorem of Art. 129 (β). Also, obviously, the ray in the pencil $J'(a)$ corresponding to the ray IJ' in the pencil $I(a')$ is the ray drawn from J' to the point at infinity on Ia, *i.e.* the tangent at J'. Similarly the tangent at I corresponds to the ray $J'I$ in the pencil $J'(a)$.

136. *A variable tangent aa' meets TP, TQ', two fixed tangents to a conic, in two ranges (a), (a') which are homographic.*

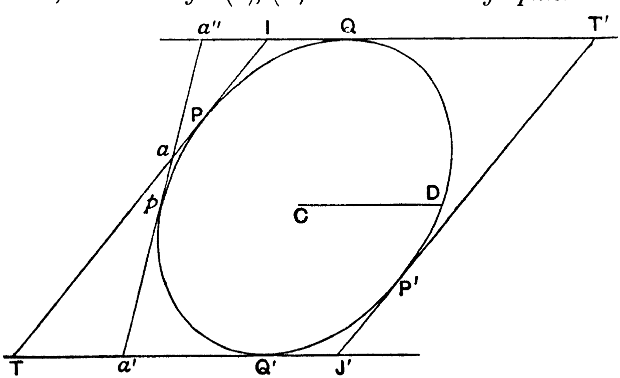

Fig. 72.

PP', QQ' are the diameters through P, Q', and CD is the semidiameter conjugate to QQ'. The tangent aa' meets the tangent at Q in a''. It is proved in Apollonius, Bk III, Prop. 42, that

$$QI \,.\, Q'T = CD^2 = Qa'' \,.\, Q'a' \quad \ldots\ldots\ldots\ldots\ldots(\text{A}).$$

Assuming this result (A), for proof of which see Milne and Davis' *Geom. Con.* Art. 134, or any other text-book, and the truth of which can be seen at once by orthogonal projection from a circle, we have

$$\begin{aligned} QI : Q'a' &= Qa'' : Q'T \\ &= Qa'' - QI : Q'T - Q'a', \\ \therefore\ J'Q' : Q'a' &= Ia'' : a'T, \text{ for } QI = J'Q' \\ &= Ia : aT, \\ \therefore\ J'Q' : J'a' &= Ia : IT, \\ \therefore\ Ia \,.\, J'a' &= IT \,.\, J'Q' = \text{const.}^{*} \end{aligned}$$

Hence the ranges (a), (a') are homographic, and we have the theorem of Art. 130 (β).

Anharmonic Properties of Points and Tangents of a Conic.

137. We will now give the complete converses of Arts. 129 (β), 130 (β), which are due to Chasles, and are two of the most important propositions in this part of the subject.

(1) *Given two homographic pencils not in perspective, the intersections of corresponding rays lie on a conic which passes through the centres of the pencils.*

(2) *Given two homographic ranges not in perspective on two given straight lines, the lines joining pairs of corresponding points envelop a conic which touches the two given lines.*

Chasles called (1) *the anharmonic property of the points of a conic*, and (2) *the anharmonic property of the tangents of a conic*, and these, with their converses given in Arts. 129, 130, he takes

* Newton's *Principia*, Bk I, Sect. V, Lemma 25.

as the fundamental propositions on which he bases his *Traité des Sections Coniques* (1865). They are first met with in Notes XV, XVI of his *Aperçu Historique* (1837), where he proves the properties for the circle, and then employs the property of Art. 21, which shews that the cross-ratio of four collinear points is unaltered by projection. In his *Géométrie Supérieure* (1852), Chap. XXV, he treats independently the locus and envelope referred to, and shews that the curve locus

(1) passes through the centres of the pencils,

(2) cannot meet a straight line in more than two points,

(3) if about two of its points rays are rotated intersecting on the curve, the rays form two homographic pencils,

(4) two such curves can be considered as homographic figures, and can be placed in perspective with each other, and can therefore always be considered as the plane section of a cone on a circular base, with similar properties for the curve envelope, and in this way he deduces that both curves are conics.

The proofs given below are taken from Chasles' *Traité des Sections Coniques*, Arts. 8 and 9, where he adds the note "L'idée de construire sur la figure même le cercle dont la courbe engendrée sera la perspective, m'a été suggérée par M. J. Delbalat."

138. *Given two homographic pencils not in perspective, the intersections of corresponding rays lie on a conic which passes through the centres of the pencils.*

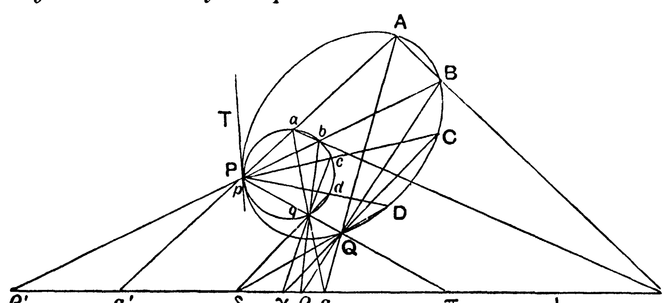

Fig. 73.

Let $P(ABC \ldots)$, $Q(ABC \ldots)$ be two homographic pencils.

It is required to shew that the locus of the points A, B, $C \ldots$ is a conic passing through the points P, Q.

Let PT be the ray in the first pencil corresponding to QP in the second, so that $P(ABCT) = Q(ABCP)$. Describe a circle touching PT at P, and let it cut PA, PB, $PC \ldots PQ$ in $a, b, c \ldots q$. Join qa, qb, $qc \ldots$, and produce them to meet the rays QA, QB, $QC \ldots$ in α, β, $\gamma \ldots$.

Now
$$\begin{aligned} Q(ABCP) &= P(ABCT) \\ &= P(abcT) \\ &= q(abcP), \end{aligned}$$

since the pencils, centres P and q, are equiangular.

And the pencils $Q(ABCP)$, $q(abcP)$ being equicross, and having a common ray QqP, are in perspective. Therefore α, β, γ, the points where corresponding rays meet, are collinear. Let L denote the line on which they meet. Then in the two triangles QAB, qab the lines joining the vertices Aa, Bb, Qq meet in P, therefore the triangles being co-polar are also co-axial, Art. 26. Therefore the sides AB, ab meet on the line joining $\alpha\beta$, *i.e.* the line L; *i.e.* the sides of the triangles QAB, qab intersect respectively in the line L; and this will still hold when the circle is rotated about L into any other position. Therefore in the new position QA, qa; QB, qb; AB, ab are co-planar, and so Aa, Bb, Qq meet in a point, the intersection of the three planes. Thus Bb, the line joining *any* two corresponding points, passes through a fixed point O (not shewn in the figure), viz. that in which Qq intersects Aa, *i.e.* the two figures are in perspective, and O, the centre of perspective, is the vertex of a cone passing through the circle and through the curve which is the locus of the points A, B, $C \ldots$. Therefore this curve, being the section of a cone by a plane, is a conic.

139. *Given two homographic ranges not in perspective on two given straight lines, the lines joining pairs of corresponding points envelop a conic which touches the two given lines.*

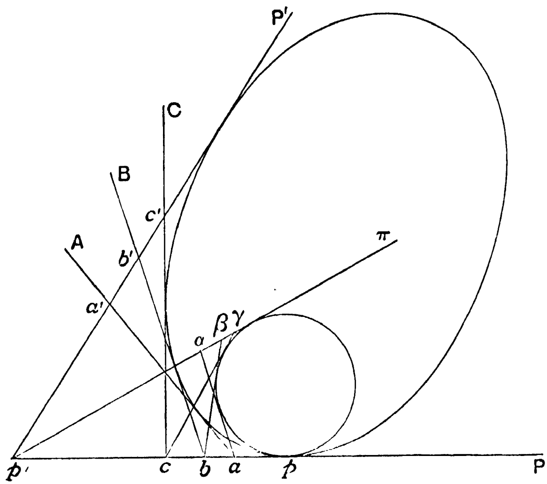

Fig. 74.

Let $(abc\ \ldots)$, $(a'b'c'\ \ldots)$ be two homographic ranges on the lines P, P'. It is required to shew that the lines aa', bb', cc' ... envelop a conic touching the lines P, P'.

Let the lines P, P' intersect in the point p', where p' is a point in the range $(a'b'c'\ \ldots)$, and let p be its corresponding point in the range $(abc\ \ldots)$. Describe a circle touching the line P at p, and to it draw the tangents $a\alpha$, $b\beta$, $c\gamma$ ... meeting the tangent $p'\pi$ in α, β, γ ....

Then the range $\quad (a'b'c'p') = (abcp)$ by hypothesis

$$= (\alpha\beta\gamma p') \text{ by Art. 130.}$$

The ranges $(a'b'c'p')$, $(\alpha\beta\gamma p')$ being equicross and having a *common point* p' are in perspective. Therefore the lines $a'\alpha$, $b'\beta$, $c'\gamma$ are concurrent.

Now rotate the circle with its tangent $p'\pi$ about the line P. For any position of the moving plane the lines $a'a$, $b'\beta$, $c'\gamma$ are concurrent, since the ranges are still in perspective, p' being the *common* point. Let them meet in O. Then with O as vertex of projection the lines aa', bb', cc' ... will be the projections of the lines $a\alpha$, $b\beta$, $c\gamma$ .... Therefore the curve enveloped by the lines aa', bb', cc' ... will be the perspective of the curve enveloped by the lines $a\alpha$, $b\beta$, $c\gamma$ .... But the latter curve is a circle. Hence the former is a plane section of a cone on a circular base, *i.e.* a conic.

140. *A conic can be drawn through five points, no three of which are collinear.*

In Fig. 73 let P, Q, A, B, C be the five points. It is required to shew that they lie on a conic.

Take $P(ABC)$ and $Q(ABC)$ as the characteristics of two homographic pencils, and draw PT in the first pencil corresponding to QP in the second, by Art. 58, so that

$$P(ABCT) = Q(ABCP).$$

Then with the same construction and demonstration as in Art. 138 it follows that the five points P, Q, A, B, C are the projections from O of the five concyclic points p, q, a, b, c. Therefore the five points P, Q, A, B, C lie on the projection of a circle, *i.e.* on a conic.

Cor. 1. Only one conic can be drawn through five given points.

Cor. 2. By taking any other point d on the circle in its original position in Fig. 73, producing dq to meet L in δ, and finding the point D where Pd intersects δQ, we can find as many more points as we please on the conic. Also, PT is obviously a tangent to the conic at P.

To obtain the tangent at Q, draw the tangent to the circle at q, meeting L in K (not shewn in the figure). Then KQ is the tangent required.

Since the tangents at any two corresponding points such as D, d intersect on L, this provides us with a simple method of drawing the tangent at any point.

The problem "to construct a conic through five points" is solved for an ellipse by Pappus, Bk VIII, Props. 13, 14, where the method depends on a property given by Apollonius, Bk III, Props. 16—23, sometimes called Newton's Theorem, viz. "The ratio of the rectangles contained by the segments of two intersecting chords of a conic is equal to the ratio of the rectangles contained by the segments of any other pair of chords parallel to them." It is needless to say that the construction is theoretical rather than practical.

For other solutions of the same problem see Newton's *Principia*, Bk I, Sect. V, Prop. 22, and Problem LIX of his *Universal Arithmetic*, and Art. 278 *infra*.

141. It only remains to consider the case when three of the points, as A, B, C, lie on a straight line. Let this meet the line through P, Q in K. The ray in the first pencil corresponding to QP in the second is now PQ, and if we describe a circle touching this ray at P, we should not obtain any point q, and the construction fails.

It should be noticed that when this case occurs, the locus consists of the two lines through A, B, C and P, Q, because the common rays PQ, QP intersect each other not only at the point K, but anywhere along the line PQ. This is evidently the case of two pencils in perspective.

142. *A conic can be drawn touching five straight lines, no three of which are concurrent.*

In Fig. 74 let P, P', aa', bb', cc' be the five lines. It is required to shew that they are tangents to a conic.

Take (abc) and $(a'b'c')$ as the characteristics of two homographic ranges, and let the point p on P correspond to p' the

intersection of the bases considered as a point on P', so that $(abcp) = (a'b'c'p')$. Then with the same construction and demonstration as in Art. 139 it follows that the five lines P, P', aa', bb', cc' are the projections from O of the five lines P, $p'\pi$, $a\alpha$, $b\beta$, $c\gamma$, which are tangents to a circle. Therefore the five given lines are tangents to the projection from O of the circle, *i.e.* a conic.

As in the previous article, if three of the lines aa', bb', cc' meet in a point u, then if P, P' meet in p', the envelope of the tangents degenerates into the two points u and p'.

143. In questions relating to the anharmonic property of tangents of a conic, if the homographic ranges are of the second order, so that the points I, J' are at a finite distance, the conic which is enveloped by the lines joining pairs of corresponding points is a central one, the centre being the mid-point of the line joining the points I, J'.

If the ranges are similar, the points I, J' are at infinity, so that the conic has its centre at infinity, and is therefore a parabola.

If the points I, J' coincide at C, the intersection of the ranges, by Art. 68 (1) the homographic equation is

$$Cm \, . \, Cm' = \text{const.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)},$$

the points of contact are at infinity, the conic is a hyperbola, the bases of the ranges are asymptotes, and (A) tells us that any tangent forms with the asymptotes a triangle of constant area.

144. In dealing with the anharmonic property of points of a conic it is important to notice that any transversal is cut by the pencils in two homographic ranges, the common points of which are evidently the points where the transversal cuts the conic.

If the ranges on the transversal are similar, one of the common points is at infinity, and the transversal is parallel to an asymptote. If in addition to being similar the ranges are superposable, the common points are both at infinity, and the transversal is an asymptote.

The tangents to the conic at the vertices of the pencils are the rays which correspond to the line joining the vertices which it must be remembered are points on the conic.

If these rays are parallel, the line joining the vertices is a diameter, and its mid-point is the centre of the conic.

If we move the pencil, vertex O', parallel to itself so that O' is made to coincide with O, any transversal will be cut by the two pencils, common vertex O, in two homographic ranges. Let e, f be their common points. Then Oe, Of being the common rays of the pencils in their new position give us the directions of the two pairs of parallel corresponding rays of the pencils in their original position, and therefore these are the directions of the points at infinity.

If the common points are real and separate, Oe and Of are parallel to the asymptotes, and the curve is a hyperbola.

If the common points are real and coincident at e, the curve is a parabola, and Oe is parallel to its axis.

If the common points are imaginary, the curve is an ellipse.

Locus ad tres et quatuor lineas.

145. We will conclude this chapter with a short account of the above locus, which in point of interest can compare with any of the problems known to the ancients, the history of which makes the study of mathematics such a fascinating subject.

In the general introduction to his *Conics* Apollonius says "The third book contains many curious theorems which are useful in the synthesis of solid loci, and in discriminating between their different cases; of which theorems the greater part and the most interesting are new, and the knowledge of these enabled me to construct completely the locus ad tres et quatuor lineas, which was not completed by Euclid, but only a small part of it, and that not satisfactorily, for it was not possible for its synthesis to be completed without the knowledge of the properties which I have discovered."

From Pappus, Bk VII, c. 36, we learn that the enunciations of the two cases of the locus were as follows:

I. *If three straight lines are given in position, and from a point straight lines are drawn to them meeting them at given angles, and if the ratio of the rectangle contained by two of these lines to the square on the third is given, the point lies on a given conic.*

II. *If there are four given straight lines, and from a point straight lines are drawn to them meeting them at given angles, and if the ratio of the rectangle contained by two of these lines to the rectangle contained by the other two is given, in this case also the point lies on a given conic.*

As Apollonius does not make any further reference to the locus, nor give a solution in so many words, it was taken for granted that his solution had been lost, and no further search seems to have been made for one in his *Conics*, fortunately for mathematics, as Ball (1888) in his *Short History of Mathematics*, p. 242, tells us that "the general theorem had baffled previous geometricians, and it was in the attempt to solve it that Descartes was led to the invention of analytical geometry" (1659). Subsequently, in 1687, a geometrical solution was given by Newton (as he thought for the first time) in his *Principia*, Bk I, Sect. V, Lemmas 17—19, where he takes first the case of a trapezium, and then of any quadrilateral, and employs Apollonius, Bk III, Props. 17, 19, 21, 23. But, as was pointed out by a writer in the *Math. Gazette*, No. 6, of October 1895, Apollonius, Bk III, Prop. 54, referred to in Art. 135, not only virtually contains the solution, but also gives us the value of the constant ratio in terms of the diameters of the conic parallel to the given lines in the case where the straight lines are drawn to them at right angles.

Thus, in Fig. 71, through P draw lines PF, PF', KPK' parallel to TJ', TI, IJ', and PL, PM, PN perpendicular to them.

Then by Art. 135, $\dfrac{C\beta \,.\, C\gamma}{Ca^2} = \dfrac{Ia \,.\, J'a'}{IJ' \,.\, IJ'}$

$$= \frac{PF}{FJ'} \cdot \frac{PF'}{F'I}$$

$$= \frac{PF}{PK'} \cdot \frac{PF'}{PK}$$

$$= \frac{PN}{PL} \cdot \frac{PN}{PM},$$

which is the locus ad tres lineas when the angles are right angles. If instead of being at right angles PL, PM, PN make angles θ_1, θ_2, θ_3 with TJ', TI, IJ', the proposition obviously still holds, but the value of the constant ratio is now $\dfrac{C\beta \,.\, C\gamma}{Ca^2} \cdot \dfrac{\sin\theta_1 \,.\, \sin\theta_2}{\sin^2\theta_3}$.

The locus ad quatuor lineas at once follows by repeated applications of the above, as shewn in Milne and Davis' *Conics*, Art. 254 (1894), where the value of the constant ratio is given in terms of the angles and the diameters parallel to the given lines.

Seeing that the locus ad quatuor lineas can be derived from the converse of the anharmonic property of the points of a conic, the anharmonic property, as we might have expected, can be readily deduced from the locus. For let $ABCD$ be a quadrilateral inscribed in a conic, P, P' any two points on the curve, and draw $P\alpha$, $P'\alpha'$ perpendiculars on AB. Then expressing twice the area of the triangle PAB in the two equivalent forms

$$P\alpha \,.\, AB = PA \,.\, PB \sin APB, \text{ \&c.},$$

we have

$$\frac{P\alpha \,.\, P\gamma \,.\, AB \,.\, CD}{P\beta \,.\, P\delta \,.\, BC \,.\, AD} = \frac{\sin APB \,.\, \sin CPD}{\sin BPC \,.\, \sin DPA} = P(ABCD) \text{ by Art. 14,}$$

and

$$\frac{P'\alpha' \,.\, P'\gamma' \,.\, AB \,.\, CD}{P'\beta' \,.\, P'\delta' \,.\, BC \,.\, AD} = \frac{\sin AP'B \,.\, \sin CP'D}{\sin BP'C \,.\, \sin DP'A} = P'(ABCD).$$

Now by the property of the locus $\dfrac{Pa \, . \, P\gamma}{P\beta \, . \, P\delta} = \dfrac{P'a' \, . \, P'\gamma'}{P'\beta' \, . \, P'\delta'}$.

Therefore $\quad P(ABCD) = P'(ABCD)$.

The importance of the locus is obvious when we notice that its two cases expressed in trilinear and quadrilinear coordinates take the well-known forms $a\beta = \kappa_1\gamma^2$ and $a\beta = \kappa_2\gamma\delta$, κ_1 and κ_2 being $\dfrac{Cc^2}{Ca \, . \, Cb}$ and $\dfrac{Cc \, . \, Cd}{Ca \, . \, Cb}$, where Ca, &c. are the semi-diameters parallel to the given lines.

We shall see in a later chapter that Desargues', Pascal's and other well-known theorems are immediate consequences of it.

EXAMPLES.

1. OL, OL' are two given lines, A, B are fixed points on OL. On OL' C, D are fixed points, and m, m' variable points. Am and Bm' meet in P. Find the locus of P

(1) when the segment mm' is of constant length,
(2) when $Cm : Dm'$ is a constant ratio,
(3) when the product $Cm \, . \, Dm'$ is constant.

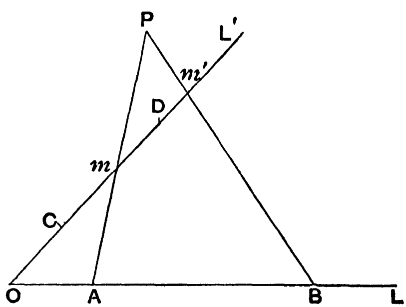

Fig. 75.

(1) As the segment mm' moves along the line OL', the ranges (m) and (m'), being superposable, are homographic.

Therefore the pencils $A\,(m)$ and $B\,(m')$ are homographic, Art. 42, and the locus of P is a conic passing through A and B, Art. 138.

Since the ranges (m), (m') made by the pencils on the transversal OL' are superposable, the line OL' is an asymptote, Art. 144.

The hyperbola will be rectangular if $mm' = AB \cos O$.

(2) Here the ranges (m), (m') are similar, but not superposable.

Therefore the locus of P is a hyperbola having one of its asymptotes parallel to OL'.

(3) By Arts. 65, 70 the ranges (m), (m') are homographic, and therefore the locus of P is a conic. To determine its species draw rays through A parallel to the rays of the pencil B. Then the conic will be a hyperbola, parabola or ellipse according as the two pencils whose common vertex is A have their common rays real and separate, real and coincident, or imaginary.

2. Shew that the locus of the vertex of an isosceles triangle whose equal sides pass through fixed points and whose base lies on a fixed straight line is a rectangular hyperbola having its centre midway between the fixed points, and one asymptote parallel to the fixed line.

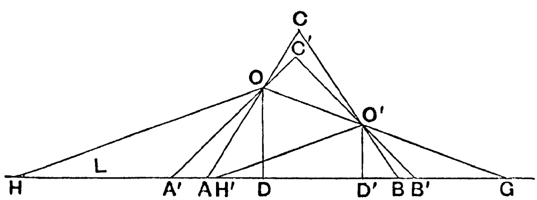

Fig. 76.

Let O, O' be the fixed points, L the fixed line, ABC, $A'B'C'$ any two positions of the triangle.

Then the angle $AOA' = A - A' = B - B' = BO'B'$.

Therefore the pencils $O(A)$ and $O'(B)$, being superposable, are homographic, and the locus of C is a conic passing through O and O'.

To find the ray of the pencil O corresponding to the ray $O'O$ in the pencil O', let OO' meet L in G. Draw OD perpendicular to L, and take $DH = DG$. Then OH is the ray required, for the angle $OHG = O'GH$.

Similarly by drawing $O'D'$ perpendicular to L we can find $O'H'$, the ray corresponding to OO'. And OH, $O'H'$, which are the tangents at O, O', are parallel, for each makes with L the angle which OO' makes with L. Therefore OO' is a diameter, and the mid-point of OO' is the centre of the conic.

By the geometry of the figure the ranges (A) and (B) are obviously similar, but not superposable, therefore one asymptote is parallel to L; and the parallel lines OD, $O'D'$ are a pair of corresponding rays, and are therefore parallel to the other asymptote. Therefore the hyperbola is rectangular.

3. The points A, B are fixed, and a moving point P lies on a fixed line L in the same plane with AB. Prove that the locus of the orthocentre of the triangle PAB is a hyperbola, one of whose asymptotes is perpendicular to AB, and the other perpendicular to L. Also shew how to draw the tangents at A and B.

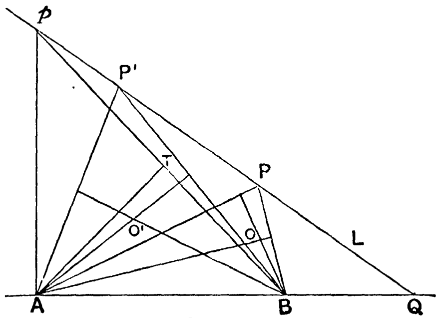

Fig. 77.

Let P, P' be two positions of the moving point, and O, O' the orthocentres of the triangles PAB, $P'AB$.

Then the angle $OAO' = PBP'$, and $OBO' = PAP'$.

Therefore the pencils $A(O)$, $B(P)$ are superposable, as are also $B(O)$, $A(P)$. And the pencils $A(P)$, $B(P)$ are homographic by Art. 45. Therefore $A(O)$ and $B(O)$ are also homographic by Art. 44; and the locus of O is a conic passing through A, B.

To find the asymptotes. When P is at infinity, AP and BP are parallel to L. Therefore the rays through A and B perpendicular to L correspond, and being parallel give the direction of an asymptote. When P is at Q the rays through A and B perpendicular to AB correspond, and give the direction of the other asymptote.

To draw the tangents at A and B. Draw Ap perpendicular to AB meeting L in p. Join Bp, and draw AT perpendicular to Bp.

Then AT is the tangent at A. Similarly we can draw the tangent at B.

4. Prove that the tangents to a parabola meet two fixed tangents in points forming two similar ranges, Art. 143. Also prove that if a straight line cuts the sides of a triangle in the points L, M, N such that the ratio of the segments is constant, it will envelop a parabola touching the sides of the triangle.

Let any tangent cut the sides of a given tangent triangle of a parabola in the points P, Q, R, and let it cut the tangent at infinity in ∞. Then by Art. 130 $(PQR\infty)$ is const., *i.e.* $\frac{PR}{P\infty} : \frac{QR}{Q\infty}$, *i.e.* $\frac{PR}{QR}$. Then use Art. 134.

5. Prove that if the corner of a rectangular piece of paper is folded down so that the sum of the edges unfolded is constant, the crease will envelop a parabola.

Let A be the corner of the rectangle $ABCD$, and let the crease cut AB in m and AD in m'. Then $mB+m'D=$const. $\therefore$ by Art. 68 (3) the ranges (m), (m') are similar. $\therefore$ &c., Arts. 136, 143.

6. Given two sides AB, AC of a triangle in position, and

(1) the sum or difference of these sides is constant, or

(2) $h \, . \, AB+k \, . \, AC=l$, where h, k, l are constant,

shew that the envelope of the base is a parabola.

By Art. 68, in each case the ranges (B) and (C) are similar. $\therefore$ &c., Arts. 136, 143.

7. Given two sides of a triangle in position and its area, shew that the envelope of the base is a hyperbola.

The area$=\frac{1}{2}AB \, . \, AC \, . \sin A$. $\therefore$ $AB \, . \, AC$ is constant. $\therefore$ by Art. 66 (1) the ranges (B) and (C) are homographic. $\therefore$ &c., Art. 143.

8. Given the base and the difference of the base angles of a triangle, the locus of the vertex is a hyperbola.

Take C, C' two positions of the vertex, AB the base.

Then the angle $CBA-CAB=C'BA-C'AB$.

$\therefore$ $CBC'=CAC'$. $\therefore$ pencil $A\,(C)=B\,(C)$.

9. Given in position two sides of a triangle, if the base subtends a constant angle at a fixed point, shew that it envelops a conic touching the two sides.

Let BC be the base, D the fixed point. The pencils $D\,(B)$, $D\,(C)$ are superposable. $\therefore$ the ranges (B), (C) are homographic, &c.

10. A perpendicular is drawn to each of two fixed tangents to a parabola at the point where it is cut by a variable tangent. Prove that their point of intersection lies on a fixed straight line.

Let the variable tangent meet the fixed tangents in m, m', and let the perpendiculars at m and m' meet in P.

By Art. 143 the ranges (m) and (m') are similar. See Chap. VI, Ex. 10.

11. Given the base and area of a triangle, shew that the locus of its orthocentre is a conic passing through the extremities of the base.

Let AB be the base. The vertex is on a line parallel to AB. Let C, C' be two positions of the vertex, P, P' the corresponding positions of the orthocentre. Then PAB is the complement of CBA, and $P'AB$ the complement of $C'BA$. $\therefore$ $PAP'=CBC'$, $\therefore$ the pencil $A\,(P)=B\,(C)$. Similarly $PBP'=CAC'$, $\therefore$ $B\,(P)=A\,(C)$. And $B\,(C)=A\,(C)$ by Art. 45, $\therefore$ $B\,(P)=A\,(P)$. $\therefore$ &c., Art. 138.

12. The line PQ subtends a right angle at each of the fixed points A and B, and the point P lies on a fixed straight line. Prove that the locus of Q is a hyperbola passing through A and B, and having one of its asymptotes perpendicular to AB, and the other perpendicular to the fixed straight line.

By Art. 45 $A(P)=B(P)$. Also $A(Q)=A(P)$, being superposable, and for a similar reason $B(Q)=B(P)$. $\therefore A(Q)=B(Q)$. $\therefore$ &c., Art. 138.

The rays through A and B perpendicular to AB correspond, and are parallel, and therefore give the direction of an asymptote, Art. 144.

13. AB, AC, two fixed tangents to a central conic, are cut by a variable tangent at m, m', and the segment mm' is divided in a constant ratio at P. The locus of P is a hyperbola having its asymptotes parallel to AB, AC, and having double contact with the given conic.

Through P draw lines parallel to the fixed tangents, meeting AB, AC in μ, μ'. Let $mP=\lambda \, . \, mm'$. Then $A\mu=(1-\lambda)\, Am$, $A\mu'=\lambda \, . \, Am'$.

Then the relation between m, m' is, by Art. 130 (β)

$$a \, . \, Am \, . \, Am' + b \, . \, Am + c \, . \, Am' + d = 0.$$

$\therefore$ by substitution the relation between μ, μ' is

$$\frac{a}{\lambda(1-\lambda)} \, . \, A\mu \, . \, A\mu' + \frac{b}{1-\lambda} \, . \, A\mu + \frac{c}{\lambda} \, . \, A\mu' + d = 0.$$

$\therefore$ the ranges (μ), (μ') are homographic, and if ∞, ∞' are points at infinity on AB, AC, the pencils $\infty'(\mu)$, $\infty(\mu')$ are homographic, and are not in perspective, because the line joining their vertices is not a common ray, since the relation between μ, μ' is of the second order.

$\therefore$ by Art. 138 the locus of P is a conic through ∞, ∞', *i.e.* a hyperbola. Moving the pencils, as in Art. 144, the common rays are evidently parallel to the fixed tangents. Also, for two positions of the variable tangent, P is a point of contact. $\therefore$ &c.

When the given conic is a parabola, the relation between m, m', and $\therefore$ also between μ, μ' is of the first order, and the pencils $\infty'(\mu)$, $\infty(\mu')$ are in perspective. Cf. Chap. VI, Ex. 10. See also Chap. VI, Ex. 25.

14. $ABCD$ is a rectangle having the side BC produced to E so that $CE=BC$. Points L and M are taken in CD, DA respectively such that $CL:CD=AM:AD$. Prove that the locus of the intersection of EL and BM is an ellipse with semi-axes CD, CB.

15. AB is a fixed diameter of a circle, mm' a movable chord perpendicular to it. Am, Bm' meet in P. Shew that the locus of P is a rectangular hyperbola.

16. If AB is a fixed chord of a circle, and CD a chord of constant length but variable position, find the locus of the intersection of the lines AD, BC.

17. AL, AL' are two given lines, and B a fixed point. Any circle passing through A and B cuts the lines in the points m, m'. Shew that the envelope of mm' is a parabola touching AL and AL'.

18. If A and B are two fixed points on a conic, and a variable tangent meets the tangents at A, B in m, m', prove that the locus of the intersection of Am', Bm is another conic.

19. Through a fixed point A on a conic two straight lines AI, AI' are drawn, S and S' are two other fixed points and P a variable point all on the conic. PS, PS' meet AI, AI' in Q, Q' respectively. Shew that QQ' passes through a fixed point.

20. BB' is the minor axis of an ellipse, and BP, BQ any two perpendicular chords through B. Shew that BP, $B'Q$ intersect on a fixed straight line.

21. AB is the base of an isosceles triangle ABC. On AC or AC produced any length AP is taken; and on BC or BC produced a length BQ is taken such that the rectangle contained by AP, BQ is equal to a constant. Shew that the locus of the intersection of AQ and BP is a conic.

22. Two sides AB, AC of a triangle are given in position, and the base BC passes through a fixed point. BP is perpendicular to AB, and CP perpendicular to AC. Shew that the locus of P is a conic. Use the method of Chap. VI, Ex. 10.

23. $ABPC$ is a parallelogram having its sides AB, AC given in position, and the diagonal BC passes through a fixed point. Shew that the locus of P is a conic.

24. Two sides of a triangle are given in position, and the circumcentre lies on a fixed straight line. Shew that the base envelops a parabola.

25. Two tangents to a conic are fixed, and two others are drawn so as to form with the first pair a quadrilateral having two opposite sides along the fixed tangents. Shew that the locus of the intersection of the diagonals of this quadrilateral is a straight line.

26. If the three sides QR, RP, PQ of a movable triangle PQR pass through the fixed points D, E, F respectively, and P lies on a fixed conic through E and F, whilst Q lies on a fixed conic through F and D, then R lies on a fixed conic through D and E.

27. TA, TB are fixed tangents to a conic, and are cut by a variable tangent in the points m, m'. Shew that the locus of the circumcentre of the triangle Tmm' is a conic.

28. A, B are two fixed points on a conic, and the variable chords Am, Bm' intersect on a fixed straight line. Shew that the locus of the intersection of Am', Bm is a conic.

29. If two triangles circumscribe a conic, their six vertices will lie on another conic.

30. If two triangles are inscribed in a conic, their six sides will touch another conic.

31. ABC is a given triangle, and PQR is a triangle of constant species inscribed in it. Shew that the sides of the latter triangle envelop three parabolas having the same focus.

32. If a polygon of constant species moves in such a manner that three of its vertices move along three fixed straight lines which are not concurrent, shew that the sides envelop parabolas all of which have the same focus.

33. The base of a triangle touches a given conic, its extremities move on two fixed tangents to the conic, and the other two sides of the triangle pass through fixed points. Find the locus of the vertex.

Let AB, AC be the two fixed tangents, D, E the fixed points, PQR any position of the triangle so that P is on AB, Q on AC, D on PR, E on QR. Since PQ is a tangent, (P) and (Q) are homographic ranges. $\therefore$ $D(P)$ and $E(Q)$ are homographic pencils, as are also $D(R)$ and $E(R)$. $\therefore$ the locus of R is a conic passing through D and E.

CHAPTER XII

PASCAL—BRIANCHON—NEWTON—MACLAURIN

146. PASCAL'S THEOREM (1640). *If a hexagon is inscribed in a conic the three points of intersection of the three pairs of opposite sides are collinear*.*

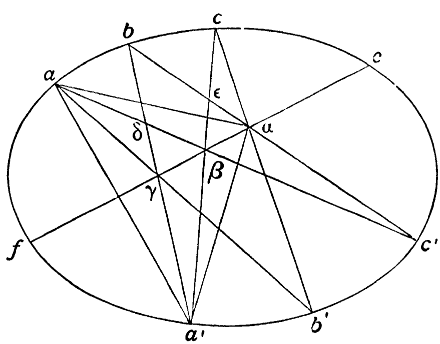

Fig. 78.

Let $ab'ca'bc'$ be a hexagon inscribed in a conic. The pairs of opposite sides are obtained by omitting one vertex in turn, and are $(ab', a'b)$; $(b'c, bc')$; $(ca', c'a)$. Let their intersections be respectively γ, α, β. These points shall be collinear.

* For the history of this theorem, which was proved by Pappus for the line-pair, and was stated, but without proof, to be true for the circle by Pascal at the age of 15, see Art. 51. For a proof by the methods of ancient geometry, applicable both to the line-pair and conic, see Appendix II.

For by Art. 129 $a(bb'c'a') = c(bb'c'a')$.

Therefore cutting these pencils by the transversals ba', bc',

$$\text{the range } (b\gamma\delta a') = (bac'\epsilon).$$

Therefore by Art. 23 the two ranges are in perspective, and consequently γa, $\delta c'$, and $\epsilon a'$ are concurrent, *i.e.* γa passes through β.

147. *Conversely, if a hexagon has the three intersections of the three pairs of opposite sides collinear, it can be inscribed in a conic.*

For in Fig. 78, taking the hexagon to be $ab'ca'bc'$ as before, the ranges $(b\gamma\delta a')$ and $(bac'\epsilon)$ have the point b common, and the lines γa, $\delta c'$, $a\epsilon$ concurrent in β. Hence the ranges are in perspective, and are therefore equicross by Art. 21.

$$\therefore\ (b\gamma\delta a') = (bac'\epsilon),$$

$$\therefore\ a(b\gamma\delta a') = c(bac'\epsilon).$$

Therefore by Art. 138 the points a, b', c, a', b, c' lie on a conic.

148. We will give another proof of this important theorem.

Taking a and a' as vertices, we have

$$\text{the pencil } a(b'cbc') = a'(b'cbc') \text{ by Art. 129,}$$
$$= a'(cb'c'b) \text{ by Arts. 3 and 16.}$$

Therefore by Art. 59 the cross-centre of the two pencils $a(b'cbc')$ and $a'(cb'c'b)$ is on the line joining the intersection of the lines $(ab', a'b')$ to the intersection of $(ac, a'c)$, *i.e.* it is on the line $b'c$.

Similarly it is on the line joining the intersection of $(ab, a'b)$ to that of $(ac', a'c')$, *i.e.* it is on the line bc'.

Therefore α, the intersection of $b'c$ and bc', is the cross-centre.

Similarly it may be shewn to be on the line joining the intersection of $(ac', a'c)$ to that of $(ab', a'b)$, *i.e.* α is on the line $\beta\gamma$.

Since a is the cross-centre of the pencils $a(b'cbc')$ and $a'(cb'c'b)$, it follows from Art. 58 that $(aa,\ a'a)$ are a pair of corresponding rays, as are also $(a'a,\ aa')$.

149. BRIANCHON'S THEOREM (1806). *If a hexagon is circumscribed about a conic, the three diagonals joining the three pairs of opposite vertices are concurrent.* (The correlative of Pascal's Theorem.)

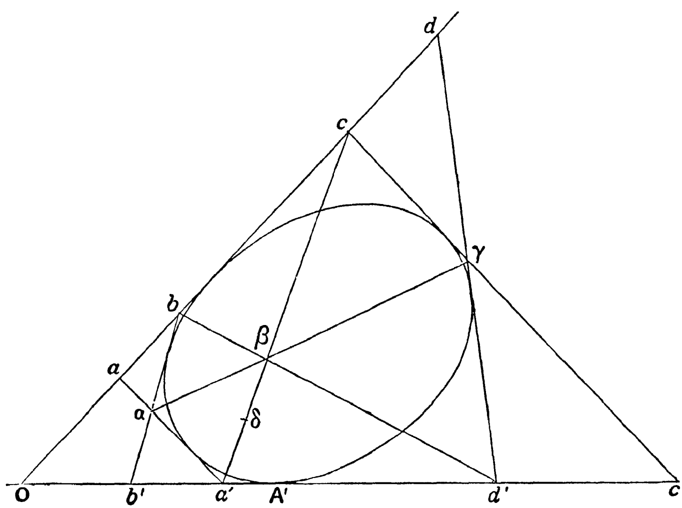

Fig. 79.

The hexagon is $abc\gamma d'a'$.

$$\text{The range } (abcd) = (a'b'c'd') \text{ by Art. 130,}$$
$$= (b'a'd'c') \text{ by Art. 3.}$$

Therefore by Art. 50 the cross-axis passes through the intersections of $(aa',\ bb')$; $(bd',\ ca')$; and $(cc',\ dd')$, *i.e.* the points α, β, γ are collinear.

Newton's Method of describing a conic (1687)*.

150. Two angles aOP, $aO'P$, of given magnitudes α, β, rotate about their vertices O, O' which are fixed. If the intersection a of two of their sides Oa, $O'a$ moves along a fixed straight line L, the intersection P of the other two sides OP, $O'P$ will describe a conic.

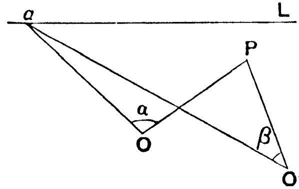

Fig. 80.

The pencils $O(a)$ and $O(P)$ are superposable, and therefore homographic. Similarly $O'(a)$ and $O'(P)$ are homographic. The pencils $O(a)$ and $O'(a)$, being in perspective, are, by Art. 45, homographic. Therefore by Art. 44 $O(P)$ and $O'(P)$ are homographic, and consequently by Art. 138 the locus of P is a conic passing through O and O'. Of course, if the point a, instead of describing the straight line L, moves along a fixed conic passing through O and O', the pencils $O(a)$ and $O'(a)$ will be homographic by Art. 129, and the locus of P will still be a conic through O, O'†.

151. MACLAURIN'S‡ THEOREM (1722). *If the sides of a triangle $aa'm$ pass through three fixed points P, Q, R, whilst two of the vertices a and a' describe straight lines OL, OL', the locus of the third vertex m is a conic.*

* *Principia*, Bk I, Sect. v, Lemma 21.

† Chasles, *Aperçu Historique*, Note xv, Sect. 9 (1837).

‡ Professor of Mathematics at Aberdeen 1717, and at Edinburgh 1725. "The one mathematician of the first rank trained in Great Britain in the 18th century," *Dictionary of National Biography*.

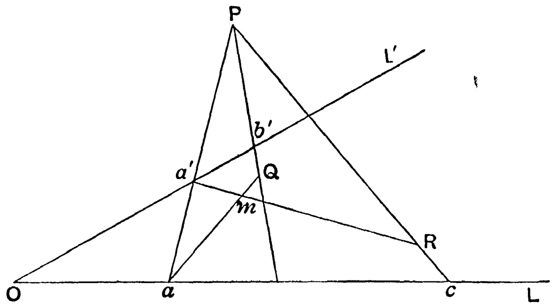

Fig. 81.

The ranges (a) and (a') are in perspective, centre P, and by Art. 41 (1) are homographic. Therefore the pencils $Q(a)$ and $R(a')$, *i.e.* $Q(m)$ and $R(m)$, are homographic. Hence by Art. 138 the locus of m is a conic through Q and R.

If PQ meets OL' in b', and PR meets OL in c, the conic will evidently pass through the points O, b', c.

152. If the points P, Q, R are in a straight line, let it meet OL, OL' in b, b'. Then this line is a common ray of the two pencils, which are consequently in perspective, and the locus of m is a straight line. See also Chap. IV, Ex. 3.

153. Maclaurin's Theorem can also be derived from Pascal's, for in Fig. 78 if we suppose a, b, c, a', c' to be five fixed points, and b' movable, and consider the triangle $\gamma b'a$, its sides pass through three fixed points, viz. a, β, c, and two of its vertices move along the fixed lines ba', bc'.

We might also derive Pascal from Maclaurin, for in Fig. 81 since by Art. 151 the locus of m is a conic through O, Q, R, b', c, m, if we consider the inscribed hexagon $Qb'OcRm$, the intersections of pairs of opposite sides obtained by omitting one vertex in turn are (Qb', cR), *i.e.* P; $(b'O, Rm)$, *i.e.* a'; (Oc, mQ), *i.e.* a; and these three points are in a straight line.

154. *If the three vertices of a triangle move on fixed straight lines, and two of its sides pass through fixed points, the third side will envelop a conic.* (Correlative of Maclaurin.)

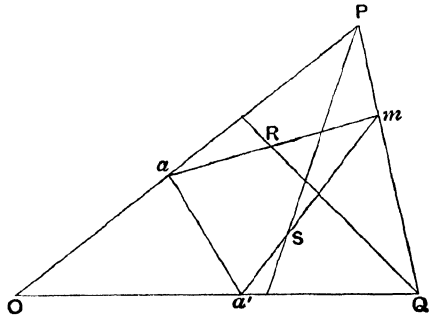

Fig. 82.

If ama' is any position of the moving triangle, the ranges (a) and (a') are obviously homographic with (m), and therefore by Art. 39 with each other. Consequently, by Art. 139, aa' envelops a conic.

By giving m different positions we see that the conic touches the five lines PS, QR, RS, OP, OQ, and it may be shewn that a similar relation exists between this proposition and Brianchon's Theorem as was shewn to hold between Maclaurin and Pascal, viz. that either of the two can be derived from the other.

155. **Common points of conic-ranges.** If two ranges of points on a conic, $abc\ldots$, $a'b'c'\ldots$, are such that the two conic-pencils formed by joining them to any other point on the curve are homographic, the ranges (*i.e.* the conic-pencils) $abc\ldots$ and $a'b'c'\ldots$ are said to be homographic, and the points where the *common rays* of the pencils cut the curve are called the *common points* of the ranges.

The student should carefully notice the difference between the homography of ranges on a conic, and that of ranges on a

straight line or lines. In the latter case the ranges are homographic when their cross-ratios are equal, but we cannot speak of the cross-ratios of ranges of points on a conic. In the latter case we always imply the cross-ratios of the conic-pencils formed by joining the points of the ranges to some point or points on the conic.

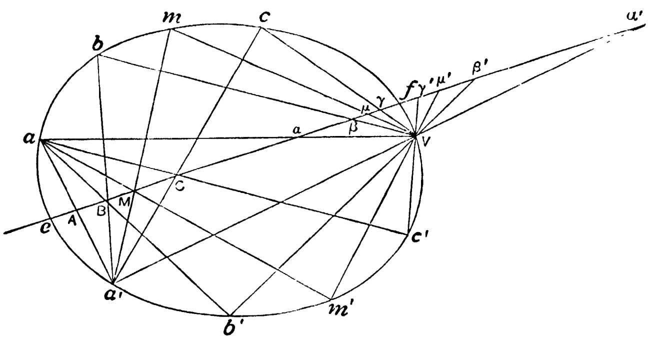

Fig. 83.

156. *Given a range of points $(abc \ldots)$ on a conic, to construct a range on the conic homographic to it.*

As in Art. 38 this can be done in an infinite number of ways; for if we take abc for the characteristic of the first range, we can take any three points a', b', c' on the conic as the characteristic of the second range.

If m is any variable point on the first range we can find its corresponding point as follows:

As in Art. 146 construct the Pascal line ef of the hexagon $ab'ca'bc'$. This will pass through B and C, the intersections of $(ab', a'b)$ and $(ac', a'c)$. Join $a'm$ meeting ef in M. Then aM will meet the conic in the required point m'.

For the pencils $a'(abcm)$ and $a(a'b'c'm')$, *i.e.* $a'(ABCM)$ and $a(ABCM)$, are homographic by Art. 45. Therefore, by Art. 129, if V is any point on the conic, $V(abcm) = V(a'b'c'm')$, *i.e.* the ranges $(abcm)$ and $(a'b'c'm')$ are homographic.

The Pascal line may be called the cross-axis of the two ranges.

157. If we suppose the point m to coincide with e, the above construction shews that m' will also coincide with e, so that e is one of the common points of the ranges. Similarly f is the other common point.

158. If the Pascal line ef is cut by the pencils $V(abcm)$ and $V(a'b'c'm')$ in the ranges $(\alpha\beta\gamma\mu)$ and $(\alpha'\beta'\gamma'\mu')$, these ranges are obviously homographic, and e, f are their common points.

159. As in Arts. 75, 80, the cross-ratio of $(\alpha\alpha'ef)$ is constant, where α, α' is any pair of corresponding points on ef, and therefore the cross-ratio of $V(aa'ef)$ is constant, where a, a' is any pair of corresponding points and V any point on the curve.

And conversely, *if e, f are two fixed points on the conic, and a, a' two variable points on the curve such that the cross-ratio $(aa'ef)$ is constant, a and a' will trace out two homographic divisions in which e, f are the common points.* Cf. Arts. 192, 201 Cor.

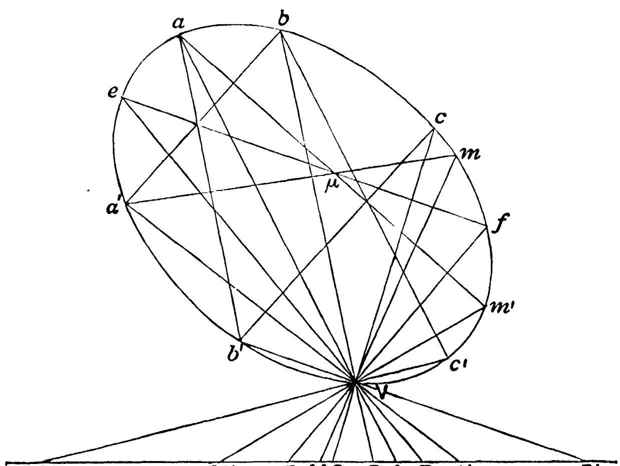

Fig. 84.

160. *On a given straight line to construct a range homographic to a given co-axial range.*

Let ABC be the characteristic of the given range. Take any three points A', B', C' on the given line to be the characteristic of the required range. Describe any conic, or circle, and on it take any point V.

Join V to the different points of the characteristics, and produce the joining lines to meet the conic in abc, $a'b'c'$. Construct the Pascal line ef of the hexagon $ab'ca'bc'$. Let M be any point on the given range. It is required to find M', the point on the given line corresponding to M.

Join MV meeting the conic in m. Join $a'm$ meeting the Pascal line ef in μ, and let $a\mu$ meet the conic in m'. Then $m'V$ will meet the given line in the required point M'.

For by Art. 156 the conic-pencils $V(abcm)$ and $V(a'b'c'm')$ are homographic, and therefore so also are the ranges $(ABCM)$ and $(A'B'C'M')$ in which these pencils are cut by the given line, the common points being E, F where the given line cuts the rays Ve, Vf. See also Arts. 82—86.

CHAPTER XIII

POLE AND POLAR. CONJUGATE POINTS AND LINES. CIRCULAR POINTS AT INFINITY. DESARGUES' THEOREM AND ITS CORRELATIVE. PROPOSITIONS RESPECTING TRIANGLES, QUADRANGLES AND QUADRILATERALS INSCRIBED IN AND CIRCUMSCRIBED ABOUT A CONIC. CONTRA-POLAR CONICS

161. *If ρ is a fixed point in the plane of a conic, and any chord is drawn through it, the locus of the fourth harmonic of ρ for the points in which the chord is cut by the conic is a straight line**.

Let $\rho aa'$ and $\rho bb'$ be any two chords through ρ. Let ap, $a'p'$ the tangents at a, a' meet in P. Let ab, $a'b'$ meet in m, and ab', $a'b$ in n, and let mn meet aa' in α and bb' in β.

Then from the quadrilateral $aa'b'b$ the ranges $(\rho a\alpha a')$, $(\rho b\beta b')$ are in perspective, centre m, and $(\rho a'\alpha a)$, $(\rho b\beta b')$ are in perspective, centre n, therefore as in Art. 118, α is the fourth harmonic of ρ for a, a' and similarly for β. We will shew that the tangents at a, a' meet on mn†.

$$\text{The pencil } a\,(abb'a') = a'\,(abb'a') \text{ by Art. } 129,$$
$$= a'\,(a'b'ba) \text{ by Arts. 3 and 16,}$$

and as the pencils have a common ray aa', they are in per-

* Apollonius, Bk III, Prop. 37.

† Lahire, Bk II, Props. 24, 27.

spective, by Art. 45, and the intersections of $(ab, a'b')$, $(ab', a'b)$ and $(aa, a'a')$, *i.e.* of the tangents ap, $a'p'$, are collinear. Hence P lies on mn.

Therefore for every chord bb' through ρ, the fourth harmonic β of ρ for the points, real or imaginary, in which the chord cuts the conic lies on the line Pa.

We have proved incidentally that the line which is the locus of the fourth harmonics passes through

(1) The cross-intersections such as $(ab', a'b)$.

(2) The intersections of tangents at the extremities of the chords through ρ.

(3) The points of contact of the tangents from ρ. These of course are imaginary if ρ is an internal point.

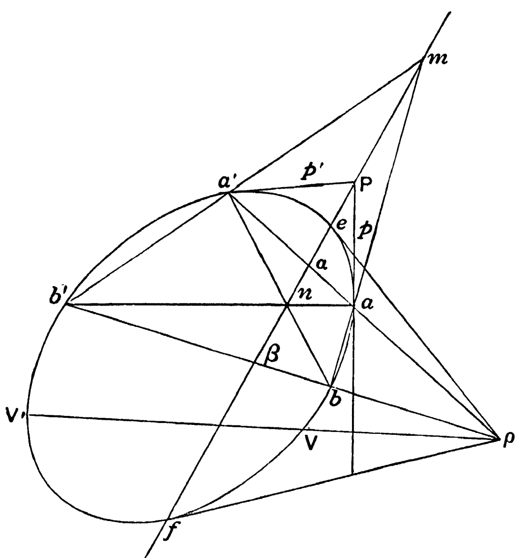

Fig. 85.

Pole and polar. Def. The point ρ and the locus of its fourth harmonic are called *pole* and *polar*.

It is obvious from the definition that every point has but one polar, and consequently every line has but one pole.

162. *If from any point P on a fixed straight line L we draw two tangents Pa, Pa′ to a conic, and a straight line L′ the fourth harmonic of L for Pa and Pa′, the line L′ will always pass through a fixed point, viz. the pole of L.* (The correlative of Art. 161*.)

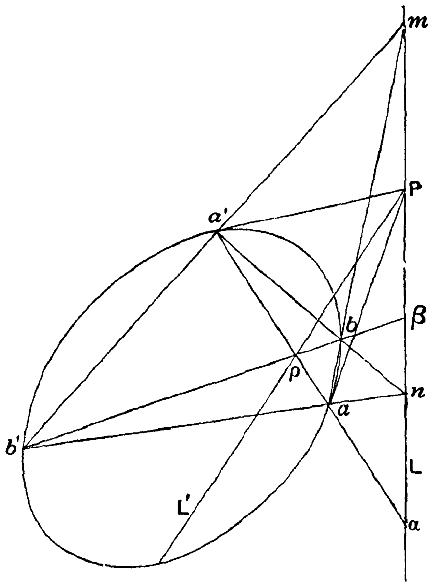

Fig. 86.

Let the chord of contact aa' meet L in α and L' in ρ. Then since $P(a\alpha\rho a')$ is a harmonic pencil, $(a\alpha\rho a')$ is a harmonic range. Therefore the polar of α passes through ρ. And the polar of P, *i.e.* aa', passes through ρ. Consequently ρ is the pole of L, and is therefore a fixed point.

* Lahire, Bk II, Props. 23, 26.

163. *The intersection of two chords is the pole of the line joining their poles.*

In Fig. 85, let aa', bb' be the chords. The pole P of the chord aa' lies on mn, as does also the pole of bb'; and the pole of mn is ρ, the intersection of the chords.

NOTE. mn obviously divides both the chords aa', bb' harmonically for ρ.

164. *Any number of rays through a fixed point intersecting a conic determine on it two sets of divisions (i.e. a conic-pencil) in involution, whose double points are the points of contact of the tangents from the fixed point.*

In Fig. 85 if the polar of the fixed point ρ meets the conic in e, f, ρe, ρf are the tangents from ρ, the polar of ρ is the axis of perspective of the pencils $a'(abc\ldots)$ and $a(a'b'c'\ldots)$, and therefore the divisions or conic-pencils $(abc\ldots)$ and $(a'b'c'\ldots)$ are homographic, and have e, f for their common points. They are also in involution, for if $\rho VV'$ is any chord through ρ, Va, $V'a'$ intersect on ef by Art. 161, as do also Va', $V'a$. Therefore ef is the axis of perspective of the pencils $V(abca')$ and $V'(a'b'c'a)$.

Therefore by Art. 24

$$\begin{aligned} V(abca') &= V'(a'b'c'a) \\ &= V(a'b'c'a) \text{ by Art. 129.} \end{aligned}$$

Therefore any transversal is cut by these homographic pencils in two ranges which by Art. 111 are in involution, and consequently, the pencils $V(abc\ldots)$ and $V(a'b'c'\ldots)$ are in involution, *i.e.* the divisions $(abc\ldots)$ and $(a'b'c'\ldots)$ are in involution, e, f being the double points.

Conversely if we have on a conic two sets of divisions forming a conic-pencil in involution, the lines joining corresponding points pass through a fixed point, and the polar of this point is the Pascal line of the system.

Therefore, to construct a conic-pencil in involution, given two pairs of conjugate elements (aa', bb'), in Fig. 85 produce aa', bb' to meet in ρ. Then any chord through ρ will give a pair of conjugate elements, and the double points of the involution are the points where the polar of ρ cuts the conic.

Conjugate points and lines.

Conjugate points.

165. Def. Two points are said to be *conjugate* for a conic when one lies on the polar of the other.

If P is a fixed point, the locus of its conjugate Q is a straight line, viz. the polar of P.

Two conjugate points cannot both lie within the conic, for the polar of each would then be outside the curve and could not pass through a point lying within it.

If P, Q are two external conjugate points, Q lies on the produced part of the chord of contact of tangents from P, and therefore PQ cannot cut the curve.

Conjugate lines.

Two lines are said to be *conjugate* for a conic when one passes through the pole of the other.

Any pair of conjugate lines through a fixed point form with the pair of tangents from the point a harmonic pencil.

Of two conjugate lines, always one, sometimes both, meet the curve.

Self-conjugate triangle.

A triangle is said to be *self-conjugate* for a conic when each vertex is the pole of the opposite side.

It follows from the above that a self-conjugate triangle has one, and only one vertex within the curve, and the side opposite to it is entirely without the curve.

166. It will be noticed that we have made frequent use of the word *conjugate*, viz. in the case of the two pairs of points in a harmonic range, in the case of two corresponding points in an

involution range, and again in the theory of pole and polar. A little consideration will shew the student that we are justified in doing so, and that the expression *two conjugate points for a conic* may be taken to imply that they possess the following properties:

(α) The two points are such that each lies on the polar of the other.

(β) The two points are harmonic conjugates for two fixed points on the line joining them, viz. the points (real or imaginary) where the line cuts the conic, and

(γ) The two points are corresponding points in two homographic co-axial ranges which together form a system in involution, the double points being the real or imaginary points in which the line joining the two conjugate points meets the curve.

Similarly the expression *two conjugate lines for a conic* implies:

(α') The two lines are such that each passes through the pole of the other. Hence a pair of conjugate diameters of a conic are conjugate lines, for each passes through the pole (at infinity) of the other. Also, any pencil of diameters is homographic to the pencil formed by their conjugates.

(β') The two lines are harmonic conjugates for two fixed lines through their point of intersection, viz. the tangents (real or imaginary) from it to the conic.

(γ') The two straight lines are corresponding rays in two concentric homographic pencils which together form a system in involution, the double rays being the tangents (real or imaginary) from the common centre of the pencils to the conic.

From the preceding articles it follows that:

(A) The pole is the conjugate of every point on its polar, and the polar is the conjugate of every line through the pole.

(B) One point, and one only (which may be at infinity), can always be found conjugate to each of two given points, viz. the

intersection of their polars; and one line, and one only (which may be at infinity), can always be found conjugate to each of two given lines, viz. the line joining their poles.

(C) The lines joining two conjugate points to the pole of the line drawn through them are the polars of the conjugate points, and are themselves conjugate lines; and the points where two conjugate lines intersect the polar of their intersection are the poles of the conjugate lines, and are themselves conjugate points, *i.e.* in each case the assemblage of lines and points form a self-conjugate triangle.

167. From Art. 166 (γ) we obtain a very important property.

In Fig. 85 since the ranges of points (P), (a) form a system in involution, they are homographic by Art. 96, and consequently so are the range (P) and the pencil $\rho\,(a)$. Hence

Given any number of poles on a straight line, the range which they form is homographic with the pencil formed by their polars for a conic.

168. If we have given a pair of lines L, L' which are not conjugate, and if any point P is taken on L, it has one and only one conjugate point P' on L', viz. the point where the polar of P meets L'; and if the points P, P' move along L and L', they will form homographic ranges by Arts. 167 and 42.

If we have given a pair of points P, P' which are not conjugate, and if any line L is drawn through P, it has one, and only one, conjugate line L' passing through P', viz. the line joining P' to the pole of L; and if the lines L, L' rotate about P and P', they will form homographic pencils, by Arts. 167 and 42.

169. We will now consider the two cases in which the line joining two conjugate points meets the conic in two (1) real, (2) imaginary points.

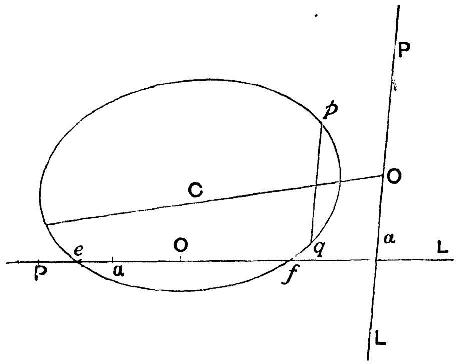

Fig. 87.

(1) Suppose we have a conic intersected by a line L in the real points e, f; and let O be the mid-point of ef. Then if P is any point on the line, and a another point on it such that $(Peaf)$ is a harmonic range, P and a are connected by the property that each lies on the polar of the other, and

$$OP \, . \, Oa = Oe^2 = Of^2. \quad \text{Art. 32.}$$

(2) If the line L does not meet the conic in real points, find C the centre of the conic, draw any chord pq parallel to L, and bisect pq by a diameter, meeting L in O. If P is any point on L, let the chord of contact of the two tangents from P meet L in a. Then P and a are conjugate points for the conic, and if E, F are the imaginary points in which L meets the conic,

$$OE^2 = OF^2 = OP \, . \, O\alpha = OQ \, . \, O\beta = \ldots .$$

In each case P and a are conjugate points of an involution range, the first system being a non-overlapping one, and the second overlapping. In both the centre O is real, and the value of the product $OP \, . \, O\alpha$ is constant and real, being positive in the first and negative in the second.

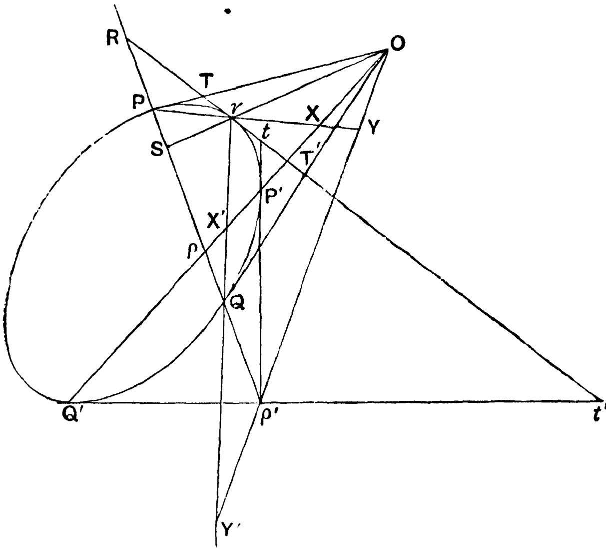

Fig. 88.

170. *To construct a triangle self-conjugate* (Art. 165) *for a conic.*

Let O be a point external to a conic. Its polar PQ meets the conic in two real points P, Q, and the other points on PQ are some of them internal and some of them external to the conic. Let ρ' be any external point on PQ. The polar of ρ' passes through O. Let it meet PQ in ρ. Then $O\rho\rho'$ is a self-conjugate triangle, having each pair of its vertices conjugate points, and each pair of its sides conjugate lines.

171. *In Fig.* 88 *if PQ, $P'Q'$ are a pair of conjugate lines which intersect within the conic, the conic-pencil* $(QP'PQ')$ *is harmonic.*

Let the chords PQ, $P'Q'$ intersect in ρ, and let the tangents at P, Q meet in O.

Then $(OP'\rho Q')$ is a harmonic range.

Therefore $Q(OP'\rho Q')$ is a harmonic pencil, and $(QP'PQ')$ is a harmonic conic-pencil.

172. *If O, ρ' are a pair of conjugate points, OP, OQ, $\rho'P'$, $\rho'Q'$ the tangents from them, $(QP'PQ')$ is a harmonic conic-pencil, and conversely if $(QP'PQ')$ is a harmonic conic-pencil, then PQ, $P'Q'$ are conjugate lines, and their poles O, ρ' are conjugate points.*

This may be stated as follows:

If two points are conjugate, their polars are conjugate lines; and if two lines are conjugate, their poles are conjugate points.

173. *If PQ, $P'Q'$ are a pair of conjugate chords, the tangents at their extremities cut any variable tangent in a harmonic range by Arts.* 131, 171.

174. *Given a fixed chord PQ, whose pole is O, and a variable point r on the conic, rP, rQ will meet any line $OP'Q'$ through O in two conjugate points.*

For by Art. 166 PQ, $P'Q'$ are conjugate lines, and by Art. 172 the conic-pencil $r(QP'PQ')$ is harmonic.

Therefore the range $(X'P'XQ')$ is harmonic, and consequently X, X' are conjugate points.

Conversely, if X, X' are any pair of conjugate points on a line through O, the lines PX, QX' intersect on the conic. In other words:

If X, X' are a pair of conjugate points, P', Q' the points where the line joining them cuts the conic, and r any point on the curve, then if rX, rX' meet the conic in P, Q, the chords PQ, $P'Q'$ are conjugate.

175. *PQ, $P'Q'$ are a pair of conjugate chords meeting in ρ, and OP, OQ, $\rho'P'$, $\rho'Q'$ the tangents at their extremities are cut by a variable tangent in T, T', t, t'. Then* (1) *Ot, Ot',* (2) *$\rho'T$, $\rho'T'$,* (3) *ρT, $\rho T'$ are pairs of conjugate lines.*

(1) By Art. 173 $(TtT't')$ is a harmonic range.

Therefore $O(TtT't')$ is a harmonic pencil, and consequently Ot, Ot' are conjugate lines.

(2) Since $\rho'(TtT't')$ is a harmonic pencil, $\rho'T$, $\rho'T'$ are conjugate.

(3) Let rP, rQ meet $O\rho'$ in Y, Y'.

Then $O\rho'$ is the polar of ρ (Art. 163), and rP is the polar of T, and therefore Y is the pole of ρT.

Similarly, since rQ is the polar of T', Y' is the pole of $\rho T'$. And since by Art. 174 Y and Y' are conjugate points, the polar of each passes through the other, *i.e.* ρT passes through Y' which is the pole of $\rho T'$, and $\rho T'$ passes through Y, which is the pole of ρT. Therefore ρT, $\rho T'$ are conjugate. Hence

The pairs of lines joining T, T' to any *point on PQ, external or internal, are conjugate.*

176. *Ot, Ot' are a pair of conjugate lines, and OP, OQ the tangents from O. A variable tangent at r meets these lines in t, t', T, T', and tP', $t'Q'$ the second tangents from t, t' intersect in ρ'. Then ρ' lies on the polar of O.*

For
$$\begin{aligned} -1 &= O(PtQt') \\ &= (TtT't') \\ &= \text{pencil of polars } r(PP'QQ'). \end{aligned}$$

Therefore the conic-pencil $(PP'QQ')$ is harmonic, and the lines PQ, $P'Q'$ are conjugate, *i.e.* ρ', the pole of $P'Q'$, lies on PQ the polar of O.

177. *The pairs of tangents to a conic from points on a given straight line determine an involution on any tangent to the conic.*

In Fig. 88 let $O\rho'$ be the given line, ρ its pole, r the point of contact of a variable tangent. Then if the tangents from any points O_1, O_2, O_3 ... on the given line meet the variable tangent at T_1, T_1'; T_2, T_2'; T_3, T_3', ..., $(\rho T_1, \rho T_1')$, $(\rho T_2, \rho T_2')$ ... are pairs

of conjugate lines by Art. 175. Therefore $\rho\,(T_1T_1', T_2T_2', \ldots)$ is an involution pencil, and $(T_1T_1', T_2T_2', \ldots)$ an involution range.

Otherwise, the chords of contact all pass through a fixed point, viz. the pole of the given line, therefore by Art. 164 they determine an involution range on the conic, &c.

178. *If two tangents are drawn to a conic, any variable tangent is divided harmonically by the two tangents, their chord of contact, and the curve.*

In Fig. 88 let OP, OQ be the given tangents, and let a variable tangent meet them in T, T', their chord of contact in R, and the curve at r. Let Or meet PQ in S. Then because the polar of O passes through R, the polar of R passes through O. But the polar of R also passes through r. Therefore the polar of R is Or.

$$\begin{aligned} \text{Therefore}\qquad -1 &= \text{the range } (RPSQ) \\ &= \text{the pencil } O\,(RPSQ) \\ &= \text{the range } (RTrT'). \end{aligned}$$

Since $(RPSQ)$ is harmonic, it follows that

Any chord of a conic is cut harmonically by any tangent and the line joining its point of contact to the pole of the chord.

Circular points at infinity.

179. The circular points have been defined in connection with orthogonal pencils in Art. 113. We will now shew how they are connected with the circle and the conic.

By Art. 114 if a segment aa' subtends a right angle at a point V, the pencil $V(aa'ii')$ is harmonic. Hence, if V, V' are any two points on the circle whose diameter is aa',

$$V(aa'ii') = V'(aa'ii').$$

Therefore by Art. 129 the six points a, a', i, i', V, V' lie on a

conic. And since this is true for all positions of V', the conic must be the circle on aa' as diameter. Hence

Every circle passes through the points i, i'.

This may also be shewn as follows. By Art. 166 (γ') for any conic a pencil consisting of pairs of conjugate lines through any point forms an involution system whose double rays are the tangents through the point, their points of contact being on the polar of the point. Now let the conic be a circle, and let the point be its centre C. Then the pencil is orthogonal, the double rays are the lines Ci, Ci' which are consequently the asymptotes to the circle, the points of contact being i, i' since the line at infinity is the polar of C, and as before we infer that all circles pass through these two points.

If the circle is of indefinitely small radius, the above property leads us to infer that it coincides with its asymptotes Ci, Ci'.

180. To obtain the converse, viz. that *any conic through* i, i' *is a circle*, draw the tangents at i, i'. Their intersection, having for its polar the line at infinity, is the centre, and the involution of conjugate diameters, having Ci, Ci' for double rays, is orthogonal, *i.e.* the conic is a circle.

181. If the angle aVa' is not a right angle, describe any circle passing through V, and cutting Va, Va' in a, a'. Then since the circle passes through i, i', the pencil $V(aa'ii')$ is constant for all points V on it. Also, if we suppose V to be fixed, and the chord aa' to vary in position whilst retaining its length, the condition that $V(aa'ii')$ is constant is equivalent to the statement that aVa' is a constant angle.

182. If the tangent at any point P of a conic meets the directrix corresponding to the focus S in the point Z, we know that the angle PSZ is a right angle, and SP, SZ are a pair of conjugate lines by Art. 166 (α'). Hence any pair of perpendicular

chords through a focus are conjugate lines, and a pencil formed of pairs of such chords, being orthogonal, is in involution, the double rays being by Art. 113 the lines joining S to i and i', and by Art. 166 (γ') these are the imaginary tangents from S to the conic, the points of contact lying on the polar of S, *i.e.* the directrix. This is sometimes expressed by saying "*The focus is a point circle having double contact with the conic along the directrix.*"

Focus. The converse of the above property is sometimes given as a definition, viz. Any point in the plane of a conic from which there can be drawn more than one pair of conjugate lines at right angles to one another is a *focus*.

Since the tangents from i, i' intersect in a focus, there are in general four foci, two being real, and two imaginary.

In the case of a parabola the circum-circle of a tangent-triangle passes through the focus, and since Si, Si' are tangents, the line joining i, i', *i.e.* the line at infinity, is a tangent.

183. If Ca, Ca' are the asymptotes of a rectangular hyperbola meeting the line at infinity in a, a', the pencil $C(aa'ii')$ is harmonic by Art. 114, and therefore by Art. 166 (β) i, i' are conjugate points for the conic.

Conversely, any conic which has i, i' for conjugate points is a rectangular hyperbola.

The triangle Cii' is evidently self-conjugate for the rectangular hyperbola.

184. FREGIER'S THEOREM. *If P is a fixed point on a conic, PQ, PR any pair of chords through P at right angles, the chord QR always passes through a fixed point on the normal at P.*

The pencil, centre P, is orthogonal, and therefore in involution. Therefore by Art. 164 the chord QR passes through a fixed point. If the angle QPR is rotated about P until R coincides with P, QR becomes the normal at P. Hence the fixed point through which QR passes lies on the normal at P.

185. *A and B are two fixed points, AP, BP lines through them conjugate for a given conic* α. *The locus of P is a conic* β *passing through A and B, and through the points where their polars meet* α.

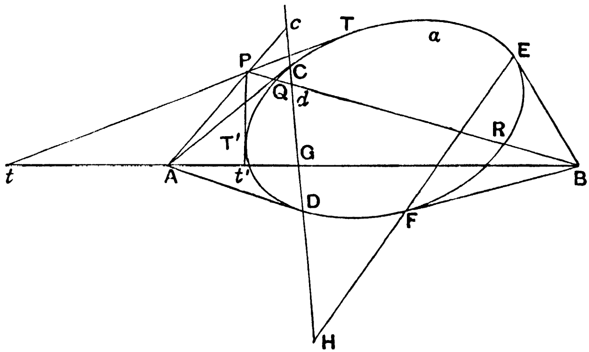

Fig. 89.

Let CD, EF be the polars of A, B, and let c, d be a pair of conjugate points on CD. Let Ac, Bd meet in P. Then since BP passes through d, the pole of Ac, AP and BP are conjugate lines. And the range (c) = the range (d), therefore the pencil $A(P)$ = the pencil $B(P)$, and by Art. 138 the locus of P is a conic passing through A and B. The locus evidently passes through the points C, D E, F, and this indirectly gives us a proof of the property of Art. 199.

Again, since P and d are conjugate points for α, the proposition might be stated as follows:

If through a fixed point B a transversal is drawn meeting a fixed line CD in d and a given conic α *in Q, R, and on it is taken the point P such that (PdQR) is harmonic, the locus of P is the conic* β. (See Art. 191.)

Also, if from any point P on β tangents PT, PT' are drawn to α, since PA, PB are conjugate lines for α these form with PT,

PT' a harmonic pencil, Art. 166 (β'), which therefore cuts AB harmonically. Hence

The locus of the intersection of tangents to a conic α *which divide a given segment* AB *of a line harmonically is the conic* β.

If the given line AB touches the given conic α, the locus β degenerates into the line AB and the line joining the points of contact of tangents to α from A and B.

If CD and EF meet in H, and if CD meets AB in G, then G, H are conjugate points for α and β, and H is the pole of AB for both conics.

Again, if on CD we take any point c, and on EF its conjugate c' for α, this pair of points will trace out two homographic ranges, and the envelope of the line joining them is a conic β' touching the lines CD, EF, and the tangents to α from A and B. This is the property of Art. 200 and may be stated as follows:

If a chord of a given conic α *is divided harmonically by the conic and by two given straight lines, its envelope is a conic* β' *touching the two given straight lines and the tangents to* α *drawn at the points where the two given lines intersect it.*

COR. If in the first part of the proposition the points A, B are the circular points at infinity, since the tangents PT, PT' divide ii' harmonically, they are at right angles by Art. 113, and the locus β becomes the director circle. If in addition the conic α is a parabola, the locus degenerates into the line at infinity and the line joining the points of contact of tangents from i, i', *i.e.* the directrix.

NOTE. The student should insert the conic β in Fig. 89, and notice that quadrilaterals can be circumscribed about α having the ends of two diagonals on β, the third diagonal being fixed, containing the given segment AB; and the fixed point H, the pole of the third diagonal, is the intersection of the other two diagonals. Hence

If α, β *are two conics such that quadrilaterals can be circumscribed about* α *and inscribed in* β, *then* β *is the locus of the intersection of tangents to* α *which divide harmonically the chord of* β *which lies on the third diagonal of any of the quadrilaterals.*

For these and other theorems of this chapter demonstrated analytically by means of the theory of invariants and covariants see Wolstenholme's *Mathematical Problems*, 3rd edition (1891), pp. 261—269. See also *infra* Chap. XIX, Exs. 22—24.

186. *If round two fixed points* ρ, ρ' *two straight lines rotate intersecting on a given conic at* α, *and cutting the conic again in* a *and* a', *the divisions* (a) *and* (a') *are homographic, and their common points are the points* e, f *where the line* $\rho\rho'$ *meets the conic*.*

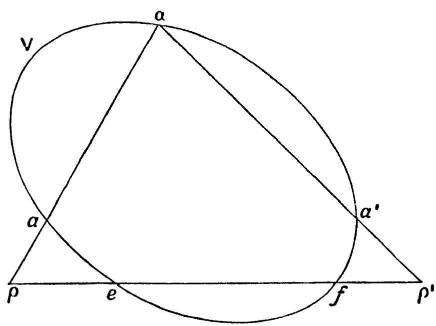

Fig. 90.

In Fig. 90 take any point V on the conic. Then since $a\alpha$ passes through the fixed point ρ, the conic-pencils $V(a)$ and $V(\alpha)$ are homographic by Art. 164. Similarly the pencils $V(a')$ and $V(\alpha)$ are homographic. Therefore by Art. 44 the pencils $V(a)$ and $V(a')$ are homographic. If α is at f, a and a' will coincide at e, so that e is a common point of the divisions (a) and (a'). Similarly f is the other common point.

* Chasles, *Sections Coniques*, Art. 229.

187. Desargues' Theorem (1593—1662). *If a quadrangle is inscribed in a conic, any transversal meets its three pairs of opposite sides and the conic in four pairs of points in involution.*

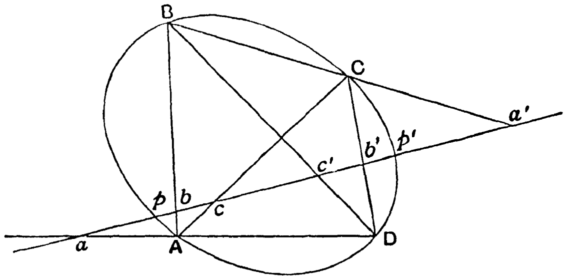

Fig. 91.

$ABCD$ is the inscribed quadrangle, pp' the transversal.

By Art. 129 $\quad A(DBpp') = C(DBpp')$,

$$\therefore (abpp') = (b'a'pp')$$
$$= (a'b'p'p) \text{ by Art. 3.}$$

Therefore by Art. 105, a, a'; b, b'; p, p' are in involution. Similarly by equating the pencils $B(ADpp')$ and $C(ADpp')$, we find that b, b'; c, c'; p, p' are in involution. Hence by Art. 104, a, a'; b, b'; c, c'; p, p' form a system in involution.

See also Art. 118 which shews that the first three pairs are in involution.

The centre O of the system can be found by the construction of Art. 102. We leave it to the student to shew that Desargues' Theorem can be obtained from Pascal's, and that each of them can be readily derived from the *locus ad quatuor lineas*, Art. 145, which is, as it were, the fons et origo of all these important properties.

188. *If a quadrilateral $aba'b'$ circumscribes a conic, and if from any point P we draw pairs of lines to its opposite vertices, and also the tangents PQ, PR, these six lines will form a pencil in involution.* (Correlative of Desargues' Theorem.)

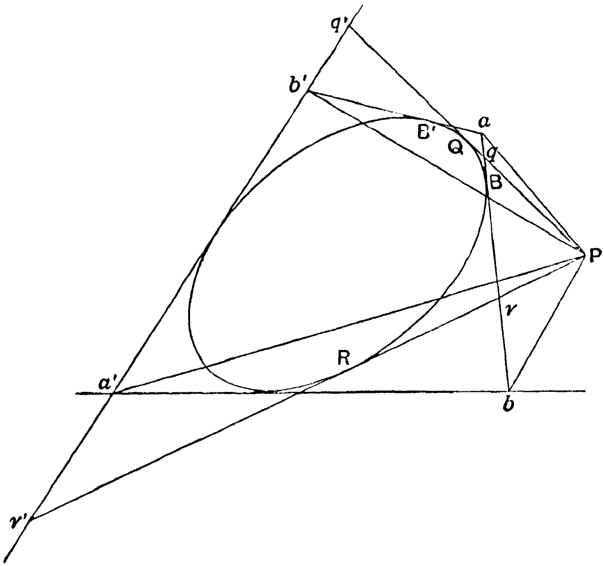

Fig. 92.

Consider the tangents ab, $a'b'$, and let them be cut by any number of tangents mm', nn' .... Then by Art. 130 the ranges $(amn \dots b)$ and $(b'm'n' \dots a')$ are homographic, as are also the pencils $P(amn \dots b)$ and $P(b'm'n' \dots a')$, and their common rays are obviously the tangents PQ, PR.

Hence
$$P(abQR) = P(b'a'QR)$$
$$= P(a'b'RQ).$$

Therefore, by Art. 105, $P(aa', bb', QR)$ is a pencil in involution.

NOTE. If ab meets $a'b'$ in c, and ab' meets $a'b$ in c', the rays Pc, Pc' belong to the same involution.

189. In Fig. 92 if the tangents $a'b$, $a'b'$ move round the conic until the point a' coincides with a, the points b, b' will coincide with the points of contact B, B' of the tangents from a, and the theorem of Art. 188 becomes

If the sides of an angle BaB' touch a conic at B, B', and if the sides of another angle QPR touch the conic at Q, R, then Pa is a double ray in each of the involution pencils $P(QR, BB')$ and $a(QR, BB')$.

190. If a quadrangle is inscribed in a conic, and a quadrilateral circumscribed about it by drawing tangents at the vertices of the quadrangle, the two figures will possess the following properties:

(1) *Their internal diagonals will intersect in the same point G, and form a harmonic pencil;*

(2) *Their third diagonals are in the same straight line, the polar of G, and their extremities form a harmonic range.*

(3) *The three diagonals of the quadrilateral and the three diagonal points of the quadrangle form the same self-conjugate triangle.*

(4) *If any transversal is drawn through any one of the three diagonal points E, F, G, the part intercepted either by the conic or by two opposite sides of either of the figures is divided harmonically by the diagonal point and its polar*.*

$ABCD$ is the quadrangle, E, F, G its diagonal points. Then since by Art. 118 (1) $(HAGC)$ is a harmonic range, FG is the fourth harmonic of FE for FA, FC. Hence, by Art. 161, Def., EF is the polar of G for the conic, and FG is the polar of E, and therefore the tangents at (A, C) and (B, D) intersect on EF. Let these tangents form the quadrilateral $prqs$.

Then p is the pole of ED, and G the pole of EF. Therefore pG is the polar of E, and is collinear with GF. In the same way it may be shewn that q lies on GF. Similarly EG passes through r and s. Therefore the two figures have their internal diagonals passing through the same point G, and the intersections E, F, t, t'

* Poncelet, *Prop. Proj.* Vol. I, § ii, cap. II, p. 97 Note, states that (1), (2) and (3) are due to Maclaurin, and (4) to Lahire.

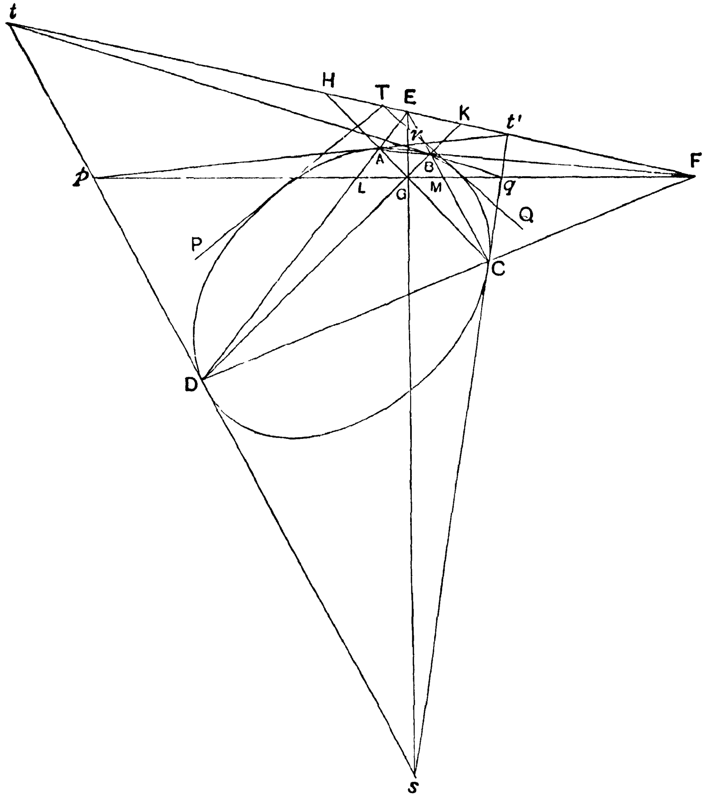

Fig. 93.

of their opposite sides lying on the same straight line. Also these four points form a harmonic range, for in the quadrilateral *prqs*, by Art. 118 (1) the diagonals *rs*, *pq* divide the third diagonal *tt'* harmonically in *E*, *F*.

Since the range formed by the points *t'*, *t*, *F*, *E* is harmonic, the pencil formed by their polars *AC*, *BD*, *pq*, *rs* is harmonic by Art. 167.

Again by Art. 118 $(ELAD)$ and $(EMBC)$ are harmonic ranges. Therefore FG is the polar of E. Similarly EG is the polar of F. Consequently EF is the polar of G, and EFG is a self-conjugate triangle.

(4) These properties follow at once by employing Desargues' Theorem and its correlative.

NOTE. If from any point T on the third diagonal we draw tangents TP, TQ, then by Art. 162 $T(PQGF)$ is a harmonic pencil. Also by Art. 118 (1) $(ACGH)$ and $(BDGK)$ are harmonic ranges. Therefore by Arts. 104, 111, $T(AC, BD, PQ)$ is an involution pencil in which TG, TF are the double rays.

191. *Given a fixed point* ρ, *a fixed line* L, *and a conic* C, *let any transversal through* ρ *meet* L *in* m, *and* C *in* a, a'. *Let*

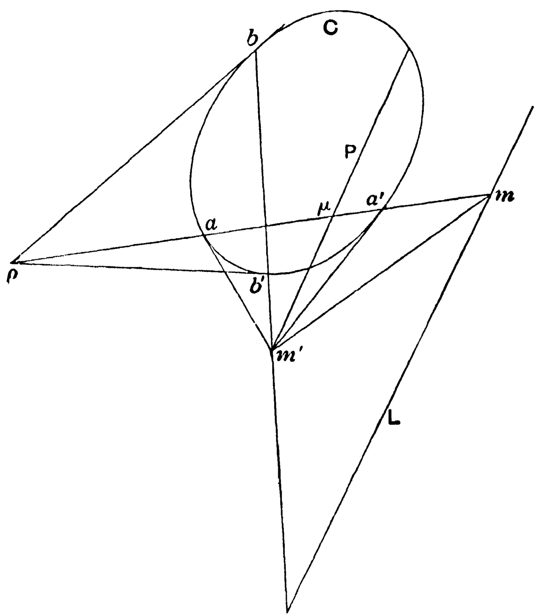

Fig. 94.

P be the pole of L, and μ the fourth harmonic of m for a, a'. As the transversal rotates about ρ, the locus of μ is a conic, and the envelope of its polar is another conic.* (See also Art. 185.)

$P\mu$ is the polar of m, therefore as m moves along L

the pencil $\rho(m)$ = the range of poles (m), by Art. 15
= the pencil of polars $P(\mu)$, Art. 167.

Therefore by Art. 138 the locus of μ is a conic through ρ, P.

In Fig. 94 let mm' the conjugate of $m\rho$ for C meet bb' the polar of ρ in m'. Then the pole of ρm, being on mm', is at m'. Also the range of poles (m') = pencil of polars $\rho(m)$, Art. 167
= the range (m), Art. 15.

Therefore by Art. 139 mm' envelops a conic which touches L and the polar of ρ.

192. *Given two homographic divisions on a conic, the rays joining the pole of the Pascal line to the common points and to any pair of corresponding points form a pencil whose cross-ratio is constant.*

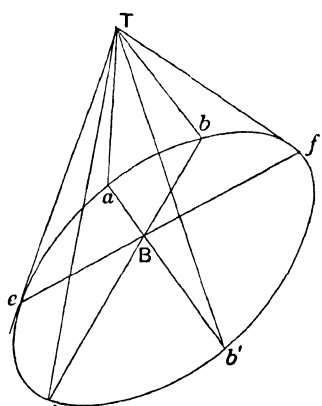

Fig. 95.

* Chasles, *Sections Coniques*, p. 136.

Let $abc\ldots$, $a'b'c'\ldots$ be two homographic divisions. By Art. 146 draw the Pascal line meeting the conic in e, f. Then by Arts. 156, 157 e, f are the common points of the divisions. Let T be the pole of ef. Then by Art. 146, B the intersection of ab', ba' lies on ef, and the polar of B passes through T, therefore T lies on the third diagonal of the quadrangle $abb'a'$. Therefore by Art. 190 Note, $T(ab', ba', ef)$ is an involution pencil, the double rays being TB and the third diagonal through T.

Therefore $\quad T(aa'ef) = T(b'bfe)$, by Art. 98

$$= T(bb'ef).$$

Conversely, if ef is a given chord of a conic, T its pole, and a, a' a pair of variable points on the conic such that $T(aa'ef)$ is constant, (a) and (a') will mark out two homographic divisions of which e, f are the common points. Cf. Arts. 159 and 201 Cor.

193. *If two triangles abc, def are inscribed in a conic C, their sides will touch another conic C'**.

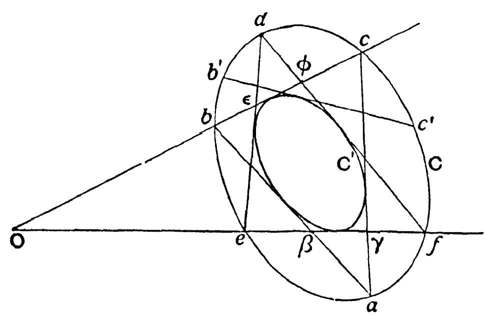

Fig. 96.

By Art. 129, $a(bcef) = d(bcef)$.

Therefore the ranges which these pencils make on ef and bc are equicross, *i.e.* $(\beta\gamma ef) = (bc\epsilon\phi)$.

Therefore, by Art. 139, $b\beta$, $c\gamma$, $e\epsilon$, $f\phi$ are tangents to a conic touching Obc and Oef.

* Brianchon, 1817.

194. *If two triangles abc, def are circumscribed about a conic C', their vertices will lie on another conic.*

In Fig. 96 by Art. 130

$$(\beta\gamma ef) = (bc\epsilon\phi),$$
$$\therefore\ a\,(\beta\gamma ef) = d\,(bc\epsilon\phi),$$
$$\therefore\ a\,(bcef) = d\,(bcef).$$

Therefore by Art. 138 the points b, c, e, f lie on a conic through a, d.

195. *If two conics C, C' are such that one triangle abc can be inscribed in C and circumscribed about C', then an infinite number of such triangles can be drawn.*

In Fig. 96 let $b'c'$ be any chord of C which touches C'. Through b', c' draw the other tangents to C', meeting in a'. Then by Art. 194 the point a' lies on the conic through the five points a, b, c, b', c', that is, it lies on C.

196. *Given a conic C and two self-conjugate triangles abc, $a'b'c'$. Their six vertices lie on a conic C', and their six sides are tangents to a conic C''.*

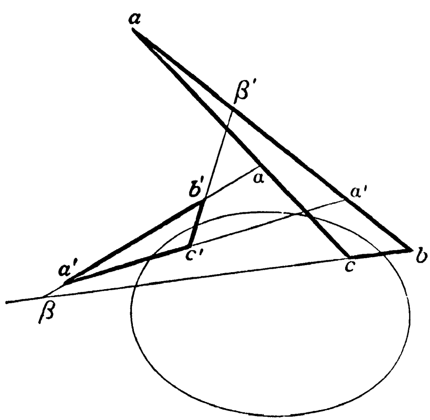

Fig. 97.

Let $a'b'$ meet ac, bc in α, β, and let ab meet $a'c'$, $b'c'$ in α', β'. Then the pencil of polars $c\,(aba'b')$

$$\begin{aligned} &= \text{range of poles } (b\alpha\beta'\alpha'), \text{ by Art. 167} \\ &= \text{pencil } c'\,(b\alpha\beta'\alpha') \\ &= \text{pencil } c'\,(bab'a') \\ &= \text{pencil } c'\,(aba'b'). \end{aligned}$$

Therefore, by Art. 138, a, b, a', b' lie on a conic through c, c'.

Again the range of poles $(ab\alpha'\beta')$ = pencil of polars $c\,(bab'a')$

$$\begin{aligned} &= \text{pencil } c\,(\beta\alpha b'a') \\ &= \text{range } (\beta\alpha b'a') \\ &= \text{range } (\alpha\beta a'b'). \end{aligned}$$

Therefore, by Art. 139, $a\alpha$, $b\beta$, $a'\alpha'$, $b'\beta'$ are tangents to a conic which touches ab and $a'b'$.

197. *If two conics C, C' are such that one triangle abc which is self-conjugate for C can be inscribed in C', then an infinite number of triangles can be inscribed in C' which are also self-conjugate for C.*

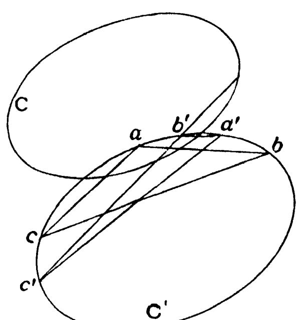

Fig. 98.

Let a' be any point on C' and let the polar of a' for C meet C' in b', c'. Then if a', b' are considered as two vertices of a triangle

self-conjugate for C, the third vertex will lie on $b'c'$. But by Art. 196 the third vertex will lie on the conic through a, b, c, a', b', *i.e.* the conic C'. Therefore the third vertex is at c'.

198. *If two conics C, C' are such that one triangle abc which is self-conjugate for C can be circumscribed about C', then an infinite number of triangles can be circumscribed about C' which are also self-conjugate for C.*

Let $b'c'$ be any tangent to C', and let a' be the pole of $b'c'$ for C, and let a tangent from a' to C' meet $b'c'$ in b'. Then if a', b' are considered as vertices of a triangle self-conjugate for C, the third vertex c' will lie on $b'c'$.

But by Art. 196 the third side $a'c'$ will touch the conic which touches ab, bc, ca, $a'b'$, $b'c'$, *i.e.* the conic C'.

199. *If a quadrilateral PRQS circumscribes a conic, its points of contact and two of its opposite vertices lie on a conic.*

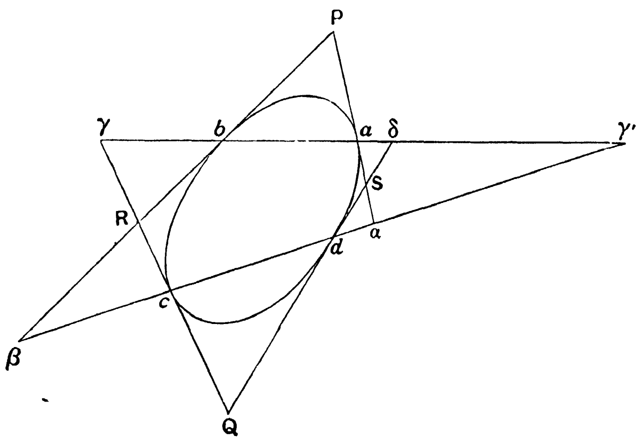

Fig. 99.

P is the pole of ab, and c is the pole of QR. Therefore γ is the pole of Pc. Similarly δ is the pole of Pd.

The pencil of polars $P(abcd)$ = range of poles $(ab\gamma\delta)$

$$= \text{pencil } Q(ab\gamma\delta)$$
$$= \text{pencil } Q(abcd).$$

Therefore by Art. 138 a, b, c, d lie on a conic through P, Q.

Similarly a, b, c, d lie on another conic which passes through R, S, and also on a third conic passing through the points (PR, QS) and (PS, QR), not shewn in the figure.

200. *If a quadrangle abcd is inscribed in a conic, the tangents at its vertices and a pair of opposite sides are tangents to another conic.*

In Fig. 99 the range of poles $(ab\gamma\delta)$ = pencil of polars $P(abcd)$

$$= \text{pencil } P(\alpha\beta cd)$$
$$= \text{range } (\alpha\beta cd).$$

Therefore $a\alpha$, $b\beta$, $c\gamma$, $d\delta$ are tangents to a conic touching ab, cd.

There are two more such conics, viz. one touching bc, ad, and one touching ac, bd.

NOTE. The properties of Arts. 199, 200 are sometimes stated thus:

If ab, cd are two chords of a conic, P, Q their poles, the six vertices of the triangles Pab, Qcd lie on one conic, and their six sides are tangents to another.

201. *If ab, cd are two chords of a conic, P, Q their poles, then if the conic-pencil $(abcd) = \lambda$,*

$$\textit{the pencil } P(abcd) = Q(abcd) = \lambda^2.$$

In Fig. 99 $\quad \lambda = \text{conic-pencil } (abcd) = c(abcd) = (ab\gamma\gamma')$,

also $\quad \lambda = d(abcd) = d(ab\gamma'\delta) = (ab\gamma'\delta)$,

$$\therefore \lambda \times \lambda = (ab\gamma\gamma') \times (ab\gamma'\delta)$$
$$= (ab\gamma\delta), \text{ by expansion, see p. 13, Ex. 3,}$$
$$= \text{pencil of polars } P(abcd) = Q(abcd), \text{ by Art. 199.}$$

COR. If $(abc\ldots)$, $(a'b'c'\ldots)$ are two homographic divisions on a conic, e, f their common points, $(aa'ef)$ is constant, and conversely, if e, f are two given points on a conic, and a, a' a pair of variable points on the curve such that $(aa'ef)$ is constant, (a) and (a') will mark out two homographic divisions of which e, f are the common points, by Art. 192. See also Art. 159.

Contra-polar conics.

202. *Two conics α, β intersect in A, B, I, I', and the poles of one of the chords as II' for α and β are T, C respectively. Then if TA is a tangent to β,*

(1) *TB will be a tangent to β,*

(2) *CA and CB will be tangents to α.*

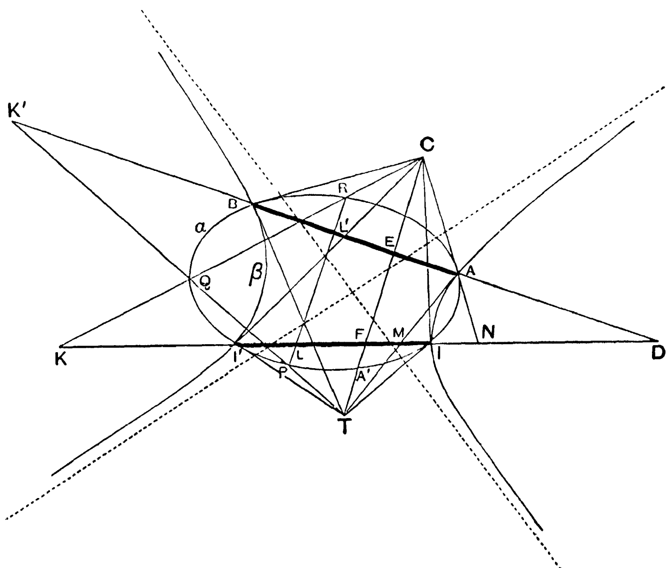

Fig. 100.

(1) Let AB, II' meet in D. Join CT meeting AB in E and II' in F. Then by Art. 163 the polar of D for α must pass through T and cut II' and AB harmonically. Similarly the polar of D for β must pass through C and divide the same two chords harmonically. Hence CT is the polar of D for both conics. And AT is the polar of A for β. Therefore T is the pole of AD for β, *i.e.* TB is the tangent to β at B.

(2) For β, C is the pole of II', and A is the pole of AT, therefore M is the pole of CA. Hence by Art. 161 $(II'MN)$ is harmonic, therefore for α, M and N are conjugate points, and T being the pole of MN, TM is the polar of N, *i.e.* NA touches α at A. And AN passes through C. Therefore CA is the tangent at A. Similarly it may be shewn that CB is the tangent at B.

203. DEF. Since the poles of a pair of common chords of α, β for one of the conics are also their poles for the other conic when the chords are taken in the contrary order, we shall call conics which are so related *contra-polar* conics, and the points C, T their *poles*.

Contra-polar conics. Poles.

From Art. 202 we obtain a simple method of describing a conic contra-polar to a given conic α. Take any point C outside α, draw a tangent CA, and join C to any two points I, I' on the curve. Then the conic which passes through A and touches CI, CI' at the points I, I' is the conic required.

204. *In two contra-polar conics the tangents at any one of the points of intersection, as A, divide the opposite chord II' harmonically.*

Let the tangents at A meet II' in M, N, and let TA meet α again in A'. Then for α, TA, II' are conjugate lines, and by Art. 171 the conic-pencil $(II'AA')$ is harmonic.

Therefore $A\,(II'AA')$ is harmonic, and therefore also $(II'NM)$, *i.e.* the tangents at A divide II' harmonically.

Similarly it may be shewn that II' is divided harmonically by the tangents at B, and AB by the tangents at I and I'.

From the property of this article these conics might be called harmotomic (on the analogy of orthotomic), a term which would help to remind the student of the connection between them and orthogonal circles, see Chap. XIX, Data 23, and Examples 51—61.

205. *In two contra-polar conics, α, β, any transversal through either of the poles is divided harmonically by the two conics.*

Let a transversal through T meet α in P, Q. We will shew that P, Q are conjugate points for β.

Join CQ meeting α again in R, and II' in K. Join RP meeting II' in L and AB in L'.

Then PQ and II' are conjugate lines for α.

$\therefore$ the pencil $R(QPII')$ is harmonic, *i.e.* $(KLII')$ is harmonic.

$\therefore$ K, L are conjugate points for β,

i.e. L is the pole of CK for β ....(1).

Again QR and AB are conjugate lines for α.

$\therefore$ the pencil $P(QRAB)$ is harmonic, *i.e.* $(K'L'AB)$ is harmonic.

$\therefore$ K', L' are conjugate points for β,

i.e. L' is the pole of TK' for β ....(2).

$\therefore$ from (1) and (2), by Art. 163, Q is the pole of LL' for β, and therefore the polar of Q passes through P. Hence P and Q are conjugate points for β, as we had to prove.

COR. If II' is a given chord of a conic β, C its pole, and if any transversal is drawn through C, and on it are taken a pair of conjugate points Q, R, then any conic through the four points Q, R, I, I' is contra-polar to the given conic β. Consequently, if we have a system of conics through four given points I, I', Q, R, a conic which passes through two of them, as I, I', and is contra-polar to one of the conics, is contra-polar to every conic of the system, and its pole for II' lies on QR, the corresponding common chord of the system.

206. If we have given four points a, b, c, d on a conic α, and if the tangents at these points form a quadrilateral whose opposite vertices are P, Q; R, S; T, U, it is clear that through the four points a, b, c, d three conics can be described contra-polar to α, viz. those having for their poles P, Q; R, S; and T, U.

207. If on a conic α we take four points forming a harmonic c. p. as $(AA'II')$ in Fig. 100, the poles being N, T, the contra-polar conic β through the four points degenerates into the lines AA', II', the tangents to β at I, I' passing through N, and those at A, A' passing through T.

NOTE. Prof. A. Lodge has pointed out to me that the relations between contra-polar conics can be readily established by means of Art. 201.

(1) If $\quad \lambda = \text{c. p. } (ABII') \text{ for } \alpha, \; \lambda^2 = C\,(ABII')$,

and if $\quad \mu = \text{c. p. } (ABII') \text{ for } \beta, \; \mu^2 = C\,(ABII')$.

$\therefore\ \lambda^2 = \mu^2$. Now since $\lambda \neq \mu$, λ must $= -\mu$, and the condition that two conics through four given points should be contra-polar is $\lambda + \mu = 0$.

(2) To prove Art. 204 we have

$$\lambda = (ABII') \text{ for } \alpha = A\,(ABII') = (MDII').$$

$$\therefore\ \frac{1}{\lambda} = (DMII') \text{ by Art. 3.}$$

$$-\lambda = \mu = (ABII') \text{ for } \beta = A\,(ABII') = (NDII').$$

$$\therefore\ -1 = -\lambda \times \frac{1}{\lambda} = (NDII') \times (DMII')$$

$$= (NMII') \text{ by expansion.}$$

(3) In the particular case when $\lambda = -1$, and the conic-pencil for α is harmonic, $\mu = +1$, and β degenerates into the two lines AB, II'.

For since $C\,(ABII') = 1$, one pair of rays CI and CI' coincide by Art. 8. Hence the tangents to β at I, I' which have to pass through C coincide with the line II'.

Similarly since $T(ABII') = 1$, TA, TB coincide, and the tangents to β at A, B must lie along AB, *i.e.* β is the two lines AB, II'. Art. 207.

208. The following useful properties in connection with conjugate points and lines are given here for convenience of reference.

Other properties will be found in Exs. 1—7, 35, 36, in which the letters have been so arranged that Fig. 101 will apply to them.

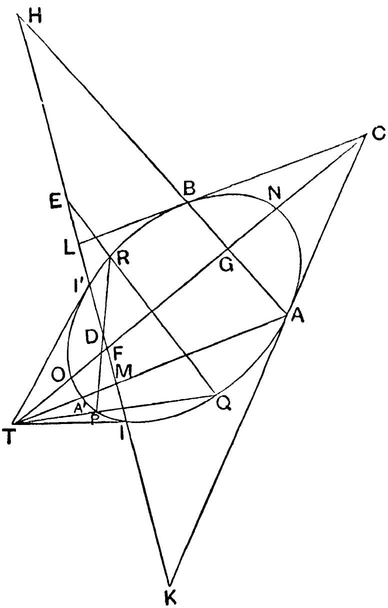

Fig. 101.

AB, II' are two chords of a conic intersecting in H; C, T their poles. CT meets AB, II' in G, F, and the curve in N, O. The tangents at A, B meet II' in K, L.

1. If TPQ is any chord through T,

 (α) PQ, II' are conjugate lines. Art. 166 (α').

 (β) the c. p. $(PQII')$ is harmonic. Art. 171.

If R is any point on the conic and RP, RQ meet II' in D, E,

 (γ) the range $(II'DE)$ is harmonic, and consequently D, E are conjugate points, TD, TE are conjugate lines and TDE is a self-conjugate triangle. Art. 174.

 (δ) Also, if D, E are two conjugate points on II', and PR is any chord through one of them, D,
 $T(II', PR, DE)$ is an involution pencil, in which TD, TE are the double rays. Art. 118.

2. (ϵ) The ranges $(ABGH)$ and $(II'FH)$ are harmonic. Art. 163.

 (ζ) $T(AB, II', KL, CH)$ is an involution pencil, in which TC, TH are the double rays. 2 (ϵ).

 (η) If TA meets II' in M, K and M are conjugate points. Art. 171.

3. (θ) If P is a variable point on the curve, the double rays of the involution pencil $P(II', AB)$ always pass through N and O. Art. 166 (A).

EXAMPLES.

[In Exs. 1—7, 35, 36 the letters are the same as in Fig. 101.]

1. II' is a given chord of a conic, T its pole. K is a point on II', and KA, KA' are two tangents cut by the variable tangent at B in the points C, C'. Then $T(II'CC')$ is harmonic. (Chap. XIX, Ex. 10.)

2. TPQ is a chord through T. The tangent at any point A meets II' in K. M is the harmonic conjugate of K for I, I'. PM meets the curve in P'. Then PA is a double line of the involution pencil $P(II'QP')$ and QP' passes through K. (Chap. XIX, Ex. 15.)

3. Given a chord II', T its pole, and any chord PR cutting II' in D. Divide PR at S so that $(PRSD)$ is harmonic, and let TS meet II' in E. Then $(II'DE)$ is harmonic. Conversely, if $(II'DE)$ is harmonic, so also is $(PRSD)$.

Also, TD, TE are the double rays of the involution pencil $T(II', PR, DE)$. (Chap. XIX, Ex. 6.)

4. II' is a chord of a conic, T its pole, D a fixed point on II'. If DPR, any chord through D, is divided harmonically at D, S, the locus of S is a straight line through T; and if this line meets II' in E, the range $(II'DE)$ is harmonic.

5. II', AB are two given chords of a conic, T, C their poles. AB meets II' in H, and $(II'FH)$ is harmonic. TPQ is a chord through T. Then II' is one of the double rays of the involution pencil $F(AB, PQ)$.

6. D, E are two conjugate points, R any point on the conic. If RD, RE meet the curve in P, Q, then PQ and DE are conjugate lines.

7. II' is a given chord of a conic, T its pole, $TA'A$ a chord meeting II' in M. The tangent at any point B meets II' in L, and V is taken on II' so that $(II'LV)$ is harmonic. AV meets the curve in W. Then AB is a double line of the involution pencil $A(II', A'W)$ and $A'W$ passes through L.

8. II' is a chord, T its pole, TAB, $TA'B'$ two chords through T. If AA', BB' meet in C, and AB', $A'B$ meet in D, the points C, D will lie on II', and divide it harmonically. (Chap. XIX, Ex. 20.)

9. AB is a chord, C its pole. Through B a straight line is drawn meeting the conic in D and AC in E. The tangent at D meets AC in F. Then $(AECF)$ is harmonic.

10. Given three tangents to a conic, draw a fourth tangent so that the part intercepted between two of the given tangents shall be bisected by the third.

[Let OT, OT' be the two given tangents, P the point of contact of the third. Join OP and produce it to meet the conic again in P'. The required tangent is parallel to the tangent at P'.]

11. AB is a chord of a conic, C its pole, T the focus. Then AB is divided harmonically by CT and the directrix.

12. Given a chord AB and its pole C, if the internal bisector of the angle ACB meets AB in D, and EF is any chord through D, then CD bisects the angle ECF.

13. In Ex. 12 the poles of all chords through D lie on the line through C perpendicular to CD.

14. P, Q are two conjugate points. P moves on a fixed straight line whose pole is R, and PQ passes through a fixed point S. Shew that the locus of Q is a conic through R, S.

15. P, Q are two conjugate points. P moves on a fixed straight line whose pole is R, and PQ subtends a constant angle at a fixed point T. Shew that the locus of Q is a conic through R, T.

16. PQ, $P'Q'$ are two conjugate chords, PR, any chord through P, meets $P'Q$ in A, QQ' in B and $Q'P'$ in C. Shew that $(RABC)$ is harmonic.

17. II' is a chord of a conic, T its pole, A a fixed point in the plane of the curve. Any chord BC through A meets II' in D, and $(BCDP)$ is harmonic. Shew that the locus of P is a conic through A and T.

[For let TP meet II' in E. Then the polar of D passes through P and T, and therefore also through E. And D, E, being conjugate points in an involution range, form two homographic divisions.

Hence $A(D)=T(E)$, *i.e.* $A(P)=T(P)$. $\therefore$ &c., Art. 138.]

18. If two chords are drawn through any point, the lines joining their extremities meet on the polar of the point and divide it harmonically.

19. In a hyperbola the portion of any tangent intercepted between the asymptotes is bisected at the point of contact.

20. Any tangent to a conic cuts the six sides of an inscribed quadrangle in an involution range of which the point of contact is a double point.

21. TP, TQ are two tangents to one branch of a hyperbola. They and the chord of contact PQ intersect an asymptote in H, K, L respectively. TH, TK, TL cut the other asymptote in H', K', L', and a parallel through T to the first asymptote cuts the second in X'. Prove that $(H'L'K'X')$ is a harmonic range, and that $H'K$, PQ, HK' are parallel.

22. The straight line PP' is the normal chord at P. The chords PQ, PQ' are equally inclined to PP'. Shew that $P'Q$ and $P'Q'$ are harmonic conjugates to PP' and the tangent at P'.

23. If through a fixed point on a conic two chords are drawn making equal angles with a fixed line, the chord joining their extremities will pass through a fixed point.

24. If a triangle is self-conjugate to a rectangular hyperbola, its circum-circle passes through the centre of the conic.

[In Art. 196 take Cii' for one of the triangles.]

25. If a triangle is inscribed in a rectangular hyperbola, its nine-point circle passes through the centre of the conic.

[Shew that the pedal triangle is self-conjugate to the conic.]

26. Shew that two conics which have the same focus and directrix may be considered as having double contact.

27. The circumcircle of a tangent triangle of a parabola passes through the focus.

[In **Art. 194** note that Sii' is a tangent triangle.]

28. O is a fixed point. A variable chord through O meets a conic in a, a', and on the chord is taken a point P such that $(OPaa')$ is constant. Shew that the locus of P is a conic.

29. The locus of the intersection of two tangents to a given conic drawn from corresponding points of an involution range is a conic passing through the double points of the involution and through the points of contact of the tangents to the given conic drawn from the double points. See Art. 185.

30. The envelope of a chord of a given conic whose extremities lie on corresponding rays of a pencil in involution is a conic touching the double rays of the involution pencil, and also touching the tangents drawn to the given conic at the points where the double rays meet it. See Art. 185.

31. Two fixed tangents to a conic meet a fixed line in A, B. A variable tangent meets the fixed tangents in Q, R, and the fixed line in S, and on it is taken a point P such that $(PQRS)$ is constant. Shew that the locus of P is a conic passing through A and B.

32. A tangent to a conic cuts two conjugate lines OP, OQ in P, Q. Shew that the other tangents from P and Q intersect on the polar of O.

33. p is a fixed point on a conic, and BCD an inscribed triangle. Any transversal through p cuts BC, CD, DB in a', b', c', and the conic in p'. Shew that $(p'a'b'c')$ is constant.

[In Fig. 91, since (pp', aa', bb', cc') is an involution range,

$$(p'a'b'c') = (pabc) = A\,(pabc) = A\,(pDBC) = \text{const.}]$$

34. If the extremities of two diagonals of a quadrilateral are conjugate points for a conic, the extremities of the third diagonal will also be conjugate; and if two of the three pairs of opposite sides of a quadrangle are conjugate lines, the third pair will also be conjugate. Hesse (1840). See Chasles, *Sect. Con.*, Arts. 133, 134.

35. II', AB are two chords of a conic, T, C their poles. TPQ is any chord through T, and AP, BQ intersect in X. Shew that the locus of X is a conic passing through A and B and contra-polar to the given conic.

[In Fig. 101 join TB meeting the conic in Y.

Then since T is a fixed point, P, Q are conjugate points in an involution

range. $\therefore (P)=(Q)$. $\therefore A(P)=B(Q)$. $\therefore$ the locus of X is a conic through A and B. It also evidently passes through I, I'.

Now when P is at B, Q is at Y. $\therefore$ the ray corresponding to AB is BY, *i.e.* the tangent at B passes through T. $\therefore$ by Art. 202, the conics are contra-polar.]

36. II' is a chord of a conic, T its pole. TPQ is any chord through T. PU, QV are two chords intersecting in X, and Y is the pole of UV. Shew that the conic through the five points U, V, X, I, I' is contra-polar to the given conic.

37. In Fig. 100 if CT meets the conic α in u, v, shew that Au, Av meet β in two points u', v' which are such that the line joining them passes through C and D.

38. In Fig. 100 if any transversal through C meets AB in W, shew that any conic through the points C, W, I, I' will be contra-polar to the conic α.

39. In Fig. 100 if a system of conics passes through the four points Q, R, I, I', and from C pairs of tangents are drawn to all the conics, the points of contact will all lie on the conic β.

CHAPTER XIV

COMMON CHORDS AND COMMON TANGENTS OF TWO OR MORE CONICS (1) PASSING THROUGH FOUR POINTS, (2) TOUCHING FOUR STRAIGHT LINES, *i.e.* OF PENCILS AND RANGES OF CONICS

Two conics circumscribing a quadrangle.

209. Two conics will in general intersect in four points which form an inscribed quadrangle.

Pair of common chords.

DEFS. The line joining any two of the points of intersection of two conics is called a *common chord* of the conics.

Of the six lines which can be so drawn, any two whose point of intersection does not lie on the conics are called a *pair* of common chords.

When we use the word pair in this restricted sense we shall print it in different type. It is obvious from Fig. 102 that there are in general three *pairs* of common chords.

Chord vertex.

The point of intersection of a *pair* of common chords we shall call a *chord vertex**.

In what follows, the three chord vertices will always be denoted by P_1, P_2, P_3.

* Suggested by Prof. A. Lodge.

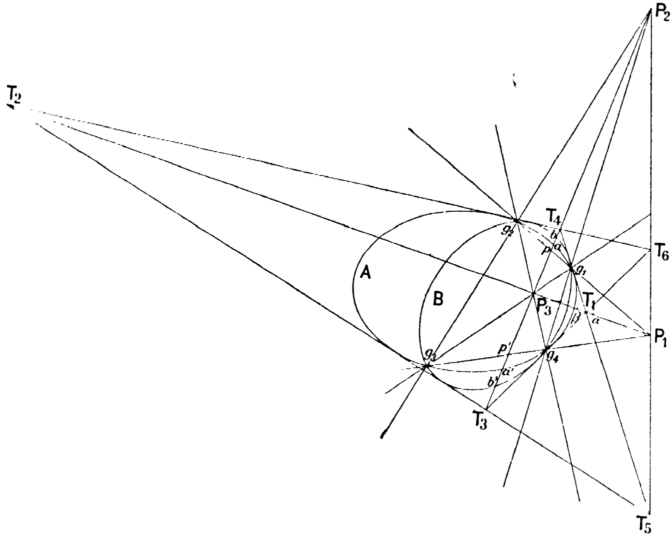

Fig. 102.

210. The distinctive property of a common chord of two conics is:

The two polars of any point on a common chord intersect on the chord.

For if P is any point on g_1g_2 a common chord of two conics A and B, and if P' is the fourth harmonic of P for g_1 and g_2, the polars of P for both A and B will pass through the point P' by Art. 161.

211. *The intersection of a* pair *of common chords has the same polar for both conics.*

For if in Fig. 102 g_1g_2 and g_3g_4 are the common chords intersecting in P_1, the two polars of P_1 considered as a point on g_1g_2 intersect on g_1g_2 by Art. 210. Considered as a point on g_3g_4, they intersect on this chord also. Hence they must coincide. Consequently,

If P_1, P_2, P_3 *are the three chord vertices,* $P_1P_2P_3$ *is a self-conjugate triangle common to both conics.*

212. *Any point* Q *which has the same polar for both conics is a chord vertex.*

For join Q to g, one of the points where the conics intersect, and produce Qg to meet the conics again in g', g'', and the common polar in Q'. Then by Art. 161, Def., $(QQ'gg')$ and $(QQ'gg'')$ are harmonic ranges.

Therefore g' and g'' coincide. Hence Q lies on one common chord. Similarly it lies on the other common chord of the *pair*, and therefore coincides with a chord vertex. Consequently

Two conics can have only one common self-conjugate triangle.

213. *The five pairs of points in which any transversal meets the two conics and the three pairs of opposite sides of the common inscribed quadrangle form a system in involution.*

By Desargues' Theorem, Art. 187.

Cor. *If the transversal is a common tangent, the points of contact are harmonic conjugates of the points where it cuts the opposite sides of the quadrangle, being the double points of the involution.*

214. *In Fig.* 102 *if the polar of* P_1 *meets the conics* A *and* B *and the pair of common chords through* P_1 *in* aa', bb', pp', *these pairs of points will form a system in involution whose double points are* P_2 *and* P_3.

By Desargues.

215. *If three conics have one chord common to all, their three other corresponding common chords are concurrent.*

Let A, B, C be the conics, and let G be the chord common to all, and let G_1 be the corresponding common chord for A and B, G_2 for A and C, and G_3 for B and C. Let G_1 and G_2 intersect

in g_1. Through g_1 draw any transversal meeting the conics in a, a'; b, b'; c, c', and meeting G and G_3 in g, g_3. Then it is evident that g_3 coincides with g_1, for by Art. 213 (aa', bb', cc') is an involution range, in which (g, g_1) and (g, g_3) are pairs of conjugate points. In other words: Given a system of conics through four fixed points, the corresponding common chord of any conic of the system with a fixed conic through two of the given points will pass through a fixed point.

Pencil of conics.

Pencil of conics. DEF. A system of conics circumscribing a quadrangle is called by Chasles a *pencil of conics**.

216. *Given a pencil of conics, any transversal will intersect them in a system of pairs of points in involution, and the double points of the system will be conjugate points for each of the conics*†. See Art. 213.

Since the double points are the harmonic conjugates of each pair of points of intersection, the polar of one passes through the other by Art. 161, Def.

By Arts. 211, 212 a pencil of conics has one, and only one, common self-conjugate triangle.

Also, since P_1 is the pole of P_2P_3, the poles of chords through P_1 lie on P_2P_3.

217. *A given pair of points are conjugate for only one conic of a pencil.*

For if P, P' are the points, the transversal PP' cuts the pencil in a range in involution of which the double points Q, Q' will not in general coincide with P, P'. By taking on PP' pairs of

* *Sect. Con.*, Art. 306.

† This proposition was enunciated for a system of three conics by Sturm, 1826.

points harmonic conjugates for P, P', we can obtain a range in involution having P, P' for double points. By Art. 35 we can find on the transversal one and only one pair of points a, a' which will divide harmonically both the segments PP', and QQ', and which will therefore be conjugate points in both involutions. Then the conic of the pencil which passes through a will also pass through a', and will have P, P' for conjugate points.

218. *In a pencil of conics the polars of any point P will all pass through the same point P'*.*

Let the polars of P for two conics A, B of the pencil intersect in P', and let PP' meet A in a, a' and B in b, b'. Then P and P' are the double points of the involution range of which the characteristic is (aa', bb'). Let a third conic C of the pencil cut the transversal PP' in c and c'. Then by Desargues' Theorem, Art. 187, c, c' belong to the involution (aa', bb'). Therefore the range $(cc'PP')$ is harmonic, and by Art. 161 the polar of P for C passes through P'. Hence

If two points are conjugate for two conics A, B, they are conjugate for the pencil of conics to which A and B belong.

If P, P' are a pair of conjugate points for a pencil, it is evident that PP' is a common tangent of the two conics of the pencil which pass through the points P, P' respectively.

From this it follows that a system of conics passing through three given points g_1, g_2, g_3 and having a given pair of conjugate points P, P', pass through a fourth fixed point g_4, viz. the fourth point of intersection of the two conics round $g_1g_2g_3$ which touch the line PP' at P and P' respectively, and therefore the system constitutes a pencil. In other words:

If three points are given on a conic, a pair of conjugate points are equivalent to a fourth point on the curve.

* Lamé, 1818.

It can also be shewn that

Two points on a conic and two pairs of conjugate points are equivalent to four points on the curve, i.e. determine a pencil of conics.

For let the lines PP' and QQ' meet in A, and on them take the points A_1, A_2 such that $(PP'AA_1)$ and $(QQ'AA_2)$ are harmonic. Then the conic through $g_1g_2AA_1A_2$ obviously belongs to the system. Again, let the line g_1g_2 meet PP' in B and QQ' in C. On PP' take B' such that $(PP'BB')$ is harmonic, and on QQ' take C' such that $(QQ'CC')$ is harmonic. Then the line-pair g_1g_2 and $B'C'$ passes through the points g_1, g_2, B, B', C, C' and therefore is a conic belonging to the system. Let these two conics intersect again in g_3, g_4. Then since P, P' are conjugate for both conics, they are conjugate for the pencil to which they belong, and similarly for Q, Q'. Hence the system forms a pencil through the points g_1, g_2, g_3, g_4.

Two points and three pairs of conjugates determine a conic.

For if we take a third pair of points R, R', by Art. 217 only one conic of the pencil will have R, R' for a pair of conjugates, unless R, R' are themselves conjugate points of the system.

219. *If a point P move along a fixed line L, its conjugate P' for a pencil of conics will describe a conic passing through the poles of L for the pencil*.*

By Art. 219 it is only necessary to consider two of the conics, A and B.

Let l_A, l_B be the poles of L for A and B. Then $P'l_A$ and $P'l_B$ are the polars of P for A and B. And by Art. 167, as P moves along L,

$$\text{the range of poles } (P) = \text{the pencil of polars } l_A(P')$$
$$\text{and also} = \text{the pencil of polars } l_B(P').$$

Therefore by Art. 44,

$$\text{the pencil } l_A(P') = \text{the pencil } l_B(P').$$

* Poncelet, *Prop. Proj.* Vol. I, Art. 370.

Therefore by Art. 138 the locus of P' is a conic passing through l_A and l_B. Now A and B are any two conics of the pencil. Therefore the locus of P' passes through the poles of L for all the conics of the pencil.

Reciprocal point. Reciprocal conic.

DEF. The point P' in which the polars of P intersect is called the *reciprocal* of the point P, and the locus of P' is called the *reciprocal conic* of L for the pencil*.

220. *The cross-ratio of the pencil formed by the polars of any point P for four conics of a pencil of conics is independent of the position of P.*

Let P' be the reciprocal of P. Let Q be any other point, Q' its reciprocal. Let L be the line PQ, l_A, $l_B \ldots$ its poles.

Then by Art. 219

$$P'(l_A l_B l_C l_D) = Q'(l_A l_B l_C l_D),$$

i.e. the cross-ratio of the pencil formed by the polars of P is equal to that of the pencil formed by the polars of any other point.

NOTE. This cross-ratio is called the cross-ratio of the four conics circumscribing the given quadrangle†.

COR. The cross-ratio of the tangents at a point of intersection of four conics of a pencil is the same as that of the tangents at each of the other common points.

221. In Art. 219 if L meets the line P_2P_3 in Q, the polars of Q will pass through P_1. Hence P_1 is a point on the reciprocal conic, and similarly for the points P_2 and P_3. Consequently

The reciprocal conic of any line for the pencil passes through the vertices of the common self-conjugate triangle.

* Poncelet, *Prop. Proj.* Vol. I, Arts. 82, 370.

† Chasles, *Traité des Sect. Con.*, Art. 325.

The reciprocal conic of L for the pencil passes through the following eleven points:

(1) The three vertices of the common self-conjugate triangle.

(2) The six points which are the harmonic conjugates of the points where L cuts the sides of the quadrangle.

(3) The double points of the involution on L determined by the pencil.

For the above reasons the reciprocal conic is sometimes called the eleven-point conic of L.

COR. If L passes through one of the vertices of the common self-conjugate triangle, as P_1, the reciprocal conic degenerates into two straight lines*, viz.

(1) The line P_2P_3, which contains the poles of L for the pencil.

(2) The fourth harmonic of L for the *pair* of common chords through P_1.

222. *The locus of the centres of the pencil of conics is a conic which passes through*

(1) *The points P_1, P_2, P_3,*

(2) *The mid-points of each of the six sides of the quadrangle,*

(3) *The double points of the involution determined by the pencil on the line at infinity*†.

Suppose L to be the line at infinity.

Two conics inscribed in a quadrilateral.

223. Fig. 102 shews us that in general two conics have four common tangents forming a circumscribing quadrilateral whose sides intersect in six points.

* Poncelet, *Prop. Proj.* Vol. I, Art. 373.

† Dr Taylor, *Geometry of Conics*, p. 284 (1881), called the locus of centres "the eleven-point conic of the quadrilateral," but it seems better to apply the term to the reciprocal conic of L.

DEF. The point of intersection of two common tangents we shall call a *tangent vertex*.

Tangent vertex.

The six tangent vertices will be denoted by the letter T with suffixes $1...6$.

As in Art. 209, we shall speak of any two opposite vertices of the circumscribing quadrilateral as a *pair* of tangent vertices.

The figure shews us that there are in general three *pairs* of tangent vertices, viz. T_1, T_2; T_3, T_4; and T_5, T_6.

224. The distinctive property of a tangent vertex is that

Any two lines through it which are conjugate for one of the conics are also conjugate for the other,

for each is the harmonic conjugate of the other for the two common tangents through the vertex, Art. 166 (β').

225. *The line joining a* pair *of tangent vertices T_1, T_2 has the same pole for both conics.*

For if it had two poles Q, Q' the line joining them would pass through T_1, being the fourth harmonic of T_1T_2 for the common tangents through T_1. Similarly it would also pass through T_2, which is impossible.

226. *Any straight line L which has the same pole l for both conics is a line joining a* pair *of tangent vertices.*

For let L meet one of the common tangents as T_1T_3 in T. Join Tl, and draw TQ, TQ' the other tangents from T to A and B. Then $T(LlT_3Q)$ is a harmonic pencil, as is also $T(LlT_3Q')$, so that TQ' coincides with TQ, *i.e.* T is a tangent vertex, and coincides with T_1 suppose. Similarly it may be shewn that L passes through T_2. Hence

The lines joining the three pairs *of tangent vertices form a common self-conjugate triangle.*

Consequently, by Art. 212,

The line joining a pair *of chord vertices passes through a* pair *of tangent vertices, and similarly, the line joining a* pair *of tangent vertices passes through a* pair *of chord vertices.*

Corresponding common chords and tangent vertices.

DEF. A *pair* of common chords are said to *correspond* to the *pair* of tangent vertices which lie on the polar of the vertex of the chords.

Thus in Fig. 102 $P_1 g_1 g_2$ and $P_1 g_3 g_4$ correspond to T_3 and T_4.

227. *The poles of any common chord lie on the line joining the corresponding tangent vertices.*

By Arts. 211, 226.

228. *If from any point P two pairs of tangents PQ, PR; PQ', PR' are drawn to the conics, the pencil $P(QR, Q'R', T_1T_2 \ldots)$ is in involution.*

By Art. 188.

229. In Art. 228, if the point P is taken at one of the points of intersection of the conics, as g_1, and if the tangents at g_1 meet T_1T_2 in α, β, the pencil $g_1(\alpha\beta T_1T_2)$ is harmonic, $g_1\alpha$, $g_1\beta$ being the double rays of the involution pencil.

Also by Art. 227 α, β are the poles of the common chord g_1g_4.

230. By Art. 214 (aa', bb', pp') is an involution range, in which the double points are evidently P_2 and P_3.

By Art. 228 $P_1(aa', bb', T_3T_4)$ is an involution pencil, in which the double rays are P_1P_2 and P_1P_3.

Therefore T_3T_4 is a pair of conjugate points in the involution range (aa', bb', pp').

231. *If three conics have a common tangent vertex, their three other corresponding tangent vertices are collinear.*

Let A, B, C be the conics, T the given tangent vertex, T_1, T_2, T_3 the corresponding tangent vertices for A and B, B and C, A and C.

Draw T_1T_2, and from any point P on it draw pairs of tangents PQ, PR, &c. to the conics, and join PT', PT'_3.

Then $P(QR, Q'R', Q''R'')$ is an involution pencil, in which PT, PT_1; PT, PT'_2; and PT', PT'_3 are pairs of corresponding rays, *i.e.* T_3 lies on the line T_1T_2.

Range of Conics.

Range of conics. Def. A system of conics inscribed in a quadrilateral is called a *range of conics.*

232. *Given a range of conics, the pairs of tangents drawn to them from the same point P will form a pencil in involution.*

See Art. 188. The double rays are the tangents at P to the conics which pass through P.

By Arts. 225, 226 a range has one, and only one, common self-conjugate triangle.

As in Art. 217 it may be shewn that

A given pair of lines are conjugate for only one conic of a range.

233. *The poles of any straight line L for the range are collinear.*

Let l_A, l_B be the poles of L for A and B, and let $l_A l_B$ meet L in P.

Then $Pl_A l_B$ and PL are conjugate lines for both A and B, and are therefore the double rays of the involution pencil formed by the pairs of tangents from P with the lines drawn from P to opposite vertices of the quadrilateral. But the tangents from P to a third conic C of the range belong to the same involution, and consequently have the same double rays, viz. their harmonic conjugates. Hence $Pl_A l_B$ passes through l_C, the pole of L for C. Hence

A pair of lines which are conjugate for two conics A, B are conjugate for the range to which A and B belong.

If OL, OL' are a pair of conjugate lines for a range, it is evident that O is a common point of the two conics of the range which touch the lines OL, OL' respectively.

The line L and the locus of its poles being conjugate lines for the range we infer that

All conics touching three given lines and having a given pair of lines conjugate touch a fourth line.

Hence, if three tangents are given and a pair of conjugate lines, the conics touch a definite fourth line, viz. the fourth common tangent of the two conics which touch the three given tangents, and of which each touches one of the two conjugate lines at their intersection.

In other words:

If three tangents are given to a conic, a pair of conjugate lines are equivalent to a fourth tangent.

We leave it to the student to shew as in Art. 218 that

(1) *A system of conics touching two given lines and having given two pairs of conjugate lines constitute a range*, and

(2) *Two tangents and three pairs of conjugate lines determine a conic.*

234. *In Art.* 233 *if the line L rotates about a fixed point P, the line which is the locus of the poles of L will envelop a conic which also touches the polars of P for the range.*

By Art. 233 it is only necessary to consider two of the conics, A and B. Let P_A, P_B be the polars of P for A and B. Then if l_A and l_B are the poles of L in any position, l_A will lie on P_A, and l_B on P_B; and as L rotates about P, the range of polars (L) is homographic to each of the ranges of poles (l_A) and (l_B), by Art. 167.

Therefore the range (l_A) = the range (l_B), by Art. 39, and by Art. 139 the line $l_A l_B$ envelops a conic touching the lines P_A and P_B.

The same reasoning shews that the conic touches the polar of P for each conic of the range.

235. *The cross-ratio of the range formed by the poles of any line L for four conics of a range is independent of the position of L.*

By Art. 233 the poles of L, viz. l_A, l_B, l_C, l_D, lie on a line L', and if M is any other line its poles, viz. m_A, m_B, m_C, m_D, lie on another line M'. Let L and M meet in the point P. Then the polar of P for A is the line $l_A m_A$, by Art. 163, and similarly for its polars for the other conics. By Art. 234 these polars are tangents to a conic which touches L' and M'.

Therefore by Art. 130 the range $(l_A l_B l_C l_D) = (m_A m_B m_C m_D)$, *i.e.* the cross-ratio of the range formed by the poles of L is equal to that of the range formed by the poles of any other line.

NOTE. Chasles called this the cross-ratio of the four conics of the range.

236. When the line L passes through P_1, its poles lie on the line P_2P_3, and the conic envelope touches P_2P_3. Hence

The conic envelope touches the sides of the common self-conjugate triangle.

237. The conic envelope of L for the range touches the following eleven lines:

(1) The three sides of the common self-conjugate triangle.

(2) The six lines obtained by joining any tangent vertex to the point P, and taking the harmonic conjugate of this line for the two common tangents through that vertex.

(3) The double rays of the involution pencil obtained by drawing pairs of tangents from P to each conic of the range.

Hence the conic envelope of L may be called "the eleven-tangent conic."

COR. If P lies on one of the sides of the common self-conjugate triangle as P_2P_3, the conic envelope degenerates into two points, viz.

(1) The point P_1, through which the polars of P pass, and

(2) The fourth harmonic of P for the *pair* of tangent vertices on P_2P_3.

238. *The locus of the centres of the range is a straight line**.

Suppose the line L in Art. 233 to be at infinity.

239. *Let $PQRS$ be a quadrangle, A, B, C its diagonal points, and let X, Y be two conjugate points for the pencil of conics circumscribing the quadrangle.*

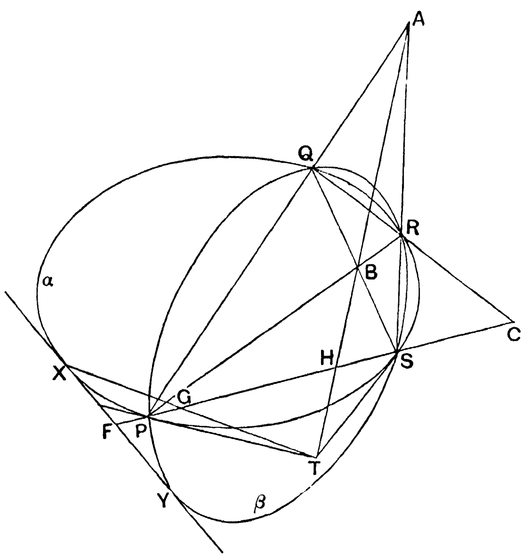

Fig. 103.

* Newton, *Princip.* Bk I, § v, Lemma 25, cor. 3.

Then $$X(PQRS) = Y(XABC)^*.$$

Of the pencil of conics, if we draw the two α, β passing respectively through the points X, Y, they will have the line XY for a common tangent by Art. 218. Then T, the pole of PS for α, lies on AB, by Art. 216. Let PS meet XY in F, XT in G, and AB in H. Then F is the pole of XT for α, and therefore, by Art. 166 (A), F and G are conjugate points ...(1).

Then the conic-pencil

$$X(PQRS) = P(PQRS) = (TABH) \quad\text{...........(2).}$$

Now by Art. 221 the eleven-point conic which is the locus of the poles of the line XY for the pencil of conics passes through the points A, B, C, X, Y, and also through the point G, since $(FGPS)$ is harmonic, by (1).

$\therefore$ the conic-pencil

$$Y(XABC) = G(XABC) = (TABH) = X(PQRS) \text{ by (2).}$$

240. *Let $PQRS$ be a quadrilateral, ABC its diagonal triangle, and let Ox, Oy be two lines conjugate for the range of conics inscribed in the quadrilateral. Let one of them, Ox, meet the lines PQ, QR, RS, SP in the points p, q, r, s, and let the other, Oy, meet BC, CA, AB in the points α, β, γ.*

Then $(pqrs) = (O\alpha\beta\gamma)$. (Correlative of Dr Milne's Theorem.)

Of the range of conics, if we draw the two a, b touching respectively the conjugate lines Ox, Oy they will have O for a common point. Let a touch the lines PQ, QR, RS, SP in the points a_p, a_q, a_r, a_s.

Then the polar of P for a passes through C. Let this polar meet Ox in x. Then since P is the pole of Ca_p, and O is the pole of Ox, therefore x is the pole of PO.

Therefore Px, PO are conjugate lines for a ....(1).

* This theorem, with a proof by analysis, was given by Dr W. P. Milne in the *Math. Gazette*, Jan. 1911, p. 386.

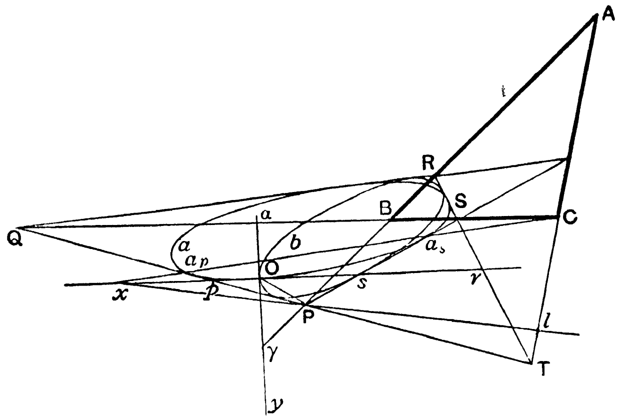

Fig. 104.

$\therefore$ the range $(pqrs)$ on the tangent Ox

$= $ conic-pencil of points of contact $(a_p a_q a_r a_s)$ Art. 131.

$= $ the range on the tangent PQ

$= (a_p QTP)$ .... (2).

Now by Art. 237 the eleven-tangent conic, which is the envelope of the polars of O for the range of conics, touches the lines BC, CA, AB, Ox, Oy, and also touches the line Px, since $P\,(xOQS)$ is harmonic, by (1).

Let Px meet BC in k and CA in l.

Then the ranges made by the four tangents Ox, BC, CA, AB on the two tangents Oy and Px are equicross, *i.e.*

$$(O\alpha\beta\gamma) = (xklP) = C\,(xklP) = (a_p QTP) = (pqrs) \text{ by (2)}.$$

EXAMPLES.

1. If two conics touch one another, the line joining the poles of their common chord passes through the point of contact.

2. Two conics touch one another at P, and T, T' are the poles of their common chord II'. If any chord PQR is drawn through P, TR, $T'Q$ will intersect on II'.

Conversely, if TR, $T'Q$ intersect on II', QR passes through P. (Chap. XIX, Ex. 28.)

3. A system of conics is described touching a given conic at a given point P, and intersecting it in two fixed points I, I'. The locus of the pole of II' for the system is a straight line passing through P. (Chap. XIX, Ex. 29.)

4. A system of conics which pass through three given points and have the poles of the line joining two of them on a fixed line have a fourth common point.

5. Given two tangents CA, CB, and two points I, I' on a conic, the locus of the pole of the common chord II' is a double line of the involution pencil $C(II', AB)$. (Chap. XIX, Ex. 30.)

6. Two conics intersect in the points A, B, I, I'. Any chord through A meets the conics in P, Q. Then $B(II'PQ)$ is constant. (Chap. XIX, Ex. 33.)

[In Examples 7—12 the reference is to Fig. 102.]

7. P, Q are the points of contact of one of the common tangents T_1T_4. The pencil $P_1(PQg_1g_4)$ is harmonic. Art. 213.

8. If the tangent P_1a meets the pair of common chords through P_2 in G, G', and the conic B in H, K, the ranges (P_1aGG') and (P_1aHK) are harmonic.

9. The tangent to the conic B at any point Q meets A in R, S, g_1g_2 in G, and g_3g_4 in G'. Then P_1Q is one of the double rays of the involution pencil $P_1(RS, g_1g_4)$. Art. 213.

10. If P, Q are the points of contact of the common tangent T_1T_4, the six points P, Q, P_1, P_2, g_1, g_4 lie on a conic having PQ, g_1g_4 for conjugate lines. See Ex. 7.

Also P, Q are conjugate points for any conic of the pencil through $g_1g_2g_3g_4$. By Desargues.

11. A straight line through T_2 meets the conic A in a, a' and B in b, b' so that the points are in the order $T_2aba'b'$. Then αa, βb meet on g_1g_4, as do also $\alpha a'$, $\beta b'$.

12. If the pole of g_1g_4 for B lies on A, the pole of g_2g_3 for B will also lie on A.

13. Three conics have a common chord the poles of which are collinear. If the polars of a point for the conics are concurrent the conics belong to a pencil.

14. A pencil of conics passes through the points A, B, I, I', and C is a fixed point in the plane. For any one of the conics T is the pole of II', and the polar of C meets TC in P. Shew that the locus of P for the pencil is a conic through C, I, I'.

15. In Art. 218, by taking the pair of conjugate points to be the circular points i, i', shew that

(1) If two conics of a pencil are rectangular hyperbolas, so are all the conics of the pencil.

(2) Two conics of the pencil touch the line at infinity at the points i, i' respectively.

(3) If A, B, C are three of the common points, the locus of the centres is the nine-point circle of the triangle ABC.

16. A is one of the common points of a pencil of conics. Any transversal through A meets three of the conics in P, Q, R, and a common chord in K. As the transversal rotates about A, $(PQRK)$ is constant, by Desargues.

17. II' is a common chord of two conics. Through T, T' its poles are drawn two chords intersecting on II'. Shew that each of the lines joining the extremities of the chords passes through a tangent vertex.

18. Shew that in Fig. 102 the points T_1, T_2, g_1, g_2, g_3, g_4 lie on a conic, *i.e.* that one of the conics of a pencil passes through a pair of tangent vertices of each pair of the conics.

19. In Fig. 102 shew that α, β are conjugate points for the conic of the pencil which passes through the points T_1, T_2.

20. Two conics touch at C and have a common chord II'. The tangent to one conic α at A meets the other conic β in B, B', and T being the pole of II' for α, TA cuts α in A'. Then CA' is one of the double rays of the involution pencil $C\ (BB', II')$.

21. If four conics are inscribed in a quadrilateral, the cross-ratio of the poles of any straight line is equal to the cross-ratio of the points of contact on each of the sides of the quadrilateral.

22. A given line L meets any conic of a pencil in a, a'. Prove that a, a' are conjugate points for the eleven-point conic corresponding to the line L.

23. If a conic γ is contra-polar to each of two conics α, β:

(1) γ will have a chord II' common to α and β,

(2) The pole of II' for γ will lie on AB, the other common chord of α and β,

(3) A and B are conjugate points for γ,

(4) γ will be contra-polar to the pencil of conics through A, B, I, I',

(5) γ will pass through two of the vertices of the common self-conjugate triangle of the pencil.

24. In Fig. 102 shew that any conic through $g_1g_4\alpha\beta$ is contra-polar to the conic of the pencil which passes through T_1, T_2. See Ex. 18.

25. In Fig. 102, $g_2\alpha$, $g_2\beta$ meet the conics A, B in a_1, b_1. Shew that the conic round $T_1T_2g_1g_2g_3g_4$ is contra-polar to the conic round $g_1g_2g_4a_1b_1$.

26. A variable conic passes through a fixed point and intersects a conic to which it is contra-polar in two fixed points. Shew that it passes through a fourth fixed point.

27. Three given conics have a common chord. Shew that the locus of a point whose polars for them are concurrent is a conic contra-polar to each of the given conics.

CHAPTER XV

HOMOLOGY

THE HOMOLOGUE OF A LINE AND CONIC. RELATIONS BETWEEN A PAIR OF COMMON CHORDS AND THE CORRESPONDING PAIR OF TANGENT VERTICES. RELATIONS BETWEEN THE FOUR CONSTANTS OF HOMOLOGY

Homologue. Centre of homology. Axis of homology. Constant of homology.

241. DEF. Given a fixed point T and a fixed line L, and any plane figure A, if through T we draw a transversal meeting A in P and L in G, and if on the transversal TP we take a point P' such that the cross-ratio $(TGPP')$ is constant $(=\lambda')$, then

The locus of the point P' is called the *homologue* of A,

The point T is called the *centre of homology*,

The line L is called the *axis of homology*,

λ' is called the *constant of homology*.

242. *The homologue of a straight line is a straight line.*

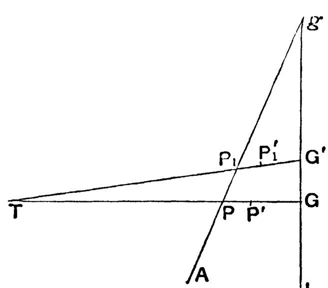

Fig. 105.

In the above Def. let A be a straight line, and let it meet L in g. Through T draw two transversals, one meeting A in P and L in G, the other meeting A in P_1 and L in G'. On TP and TP_1 take points P' and P_1' such that

$$\lambda' = (TGPP') = (TG'P_1P_1'), \text{ by Art. 9.}$$

Then by Art. 41 (2) these ranges, being homographic and having T a *common* point, are in perspective, and consequently $P'P_1'$ passes through the fixed point g. Therefore the homologue of A is a straight line passing through g.

If the line A is at infinity, its homologue is evidently a line parallel to L.

243. *The homologue of a conic is a conic.*

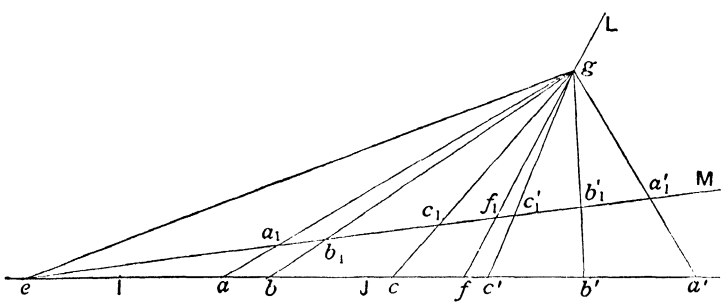

Fig. 106.

LEMMA. In Art. 74 we proved that if $abc\ldots$, $a'b'c'\ldots$ are two homographic ranges on the same straight line, and e, f their common points, the cross-ratio $(aea'f)$ is constant, where (a, a') are any pair of corresponding points. Conversely, if e, f are two fixed points on a straight line, and $abc\ldots$, $a'b'c'\ldots$ two rows of points on the same line such that $(aea'f)$ is constant, (a, a') being any pair of corresponding points, the two rows are homographic and e, f are their common points.

Through one of the common points f draw any straight line L, and from any point g on it draw rays to the points of the rows, and from e draw a transversal M meeting these rays in the points $a_1 b_1 c_1 \ldots$, $a_1' b_1' c_1' \ldots$, and meeting L in f_1.

Then by Art. 41 (1) the systems on the two transversals through e are homographic, being in perspective, centre g. Also by Art. 39 the rows $a_1 b_1 c_1 \ldots$, $a_1' b_1' c_1' \ldots$ are homographic, and e, f_1 are their common points.

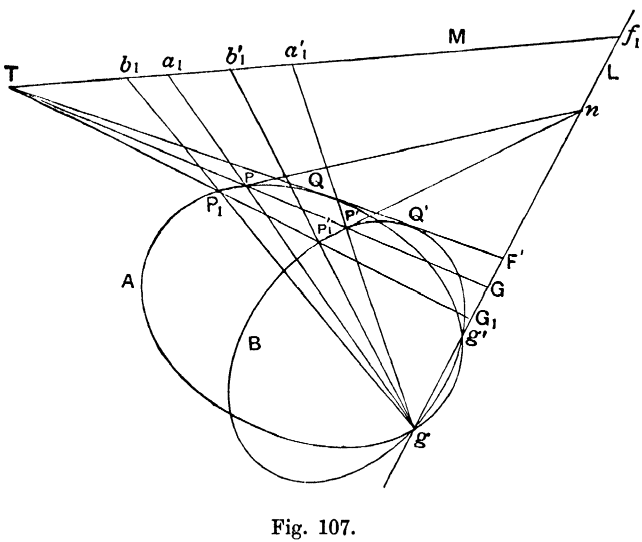

Fig. 107.

Now let A be a conic, L the axis of homology cutting A in g, g', and T the centre of homology, and let M be a fixed transversal through T.

Through T draw any transversal meeting A in P, and L in G, and on this transversal TPG take a point P' such that

$$(TGPP') = \text{a constant } \lambda', \text{ by Art. 9.}$$

Then we will prove that as the transversal rotates about T, the locus of P' is a conic passing through the points g, g'.

Through T draw any other transversal meeting the conic A in P_1, and L in G_1, and on it take the point P_1' such that

$$(TG_1P_1P_1') = \lambda'.$$

Then by Art. 41 (2) the two ranges $(TGPP')$ and $(TG_1P_1P_1')$, their cross-ratios being equal, and their point of intersection T being a *common* point, are in perspective, the centre of perspective lying on the line L.

Suppose now we draw a system of transversals through T, and form on each a range $= \lambda'$ as above. Join g to the two series of points $PP_1P_2\ldots$, $P'P_1'P_2'\ldots$. Also join gT, and let the pencil, centre g, which is thus formed be cut by the transversal M in the points $a_1b_1c_1\ldots$, $a_1'b_1'c_1'\ldots$, T, f_1.

Then by Art. 40 (2) $\lambda' = (Tf_1a_1a_1') = (Tf_1b_1b_1') = \ldots$.

Therefore by the above Lemma $a_1b_1c_1\ldots$ and $a_1'b_1'c_1'\ldots$ are two homographic rows of which T, f_1 are the common points.

$$\therefore\ (a_1b_1c_1\ldots) = (a_1'b_1'c_1'\ldots),$$

$$\therefore\ g(a_1b_1c_1\ldots) = g(a_1'b_1'c_1'\ldots),$$

$$\therefore\ g(PP_1P_2\ldots) = g(P'P_1'P_2'\ldots).$$

In a similar manner we can shew that

$$g'(PP_1P_2\ldots) = g'(P'P_1'P_2'\ldots).$$

But $\quad g(PP_1P_2\ldots) = g'(PP_1P_2\ldots)$, by Art. 129.

$$\therefore\ g(P'P_1'P_2'\ldots) = g'(P'P_1'P_2'\ldots), \text{ by Art. 44.}$$

Therefore by Art. 138 the locus of P' is a conic passing through g and g'. We will call this conic B.

244. It will be noticed that in the course of Art. 243 we incidentally proved the following property:

If through the centre of homology we draw two transversals TPP' and TP_1P_1', each meeting the conics in four points $PP'pp'$ and $P_1P_1'p_1p_1'$, the lines joining corresponding pairs of these points such as PP_1, $P'P_1'$ will meet on the axis of homology.

If the transversal TP_1 rotate into the position TP, the lines P_1P and $P_1'P'$ become the tangents at P and P', which therefore intersect on gg'.

If the transversal through T touches A at Q and meets B in Q' and L in F', then since the tangents at Q and Q' intersect on L, viz. at F', it follows that the tangent at Q is also the tangent at Q'. Hence T is the point of intersection of a pair of common tangents and gg' is one of the corresponding common chords. Consequently if we have two conics, one of them is the homologue of the other, a tangent vertex being the centre of homology, and one of the corresponding common chords the axis of homology.

Relations between a pair of common chords and the corresponding pair of tangent vertices.

245. The object of the next two articles is to prove the following propositions:

(1) *If from any point on a common chord we draw the four tangents to the two conics, the straight lines joining the points of contact on the first conic to the points of contact on the second conic will pass by pairs through the two tangent vertices which correspond to the common chord.*

(2) *If through a tangent vertex we draw any transversal meeting the conics in four points, and draw the tangents at these points, the pair of tangents to the one conic will meet the tangents to the other in four points which lie by pairs on the pair of common chords corresponding to the tangent vertex.*

246. *If from any point on the axis of homology four tangents are drawn, one pair of the lines joining the points of contact pass through one of the corresponding tangent vertices, and another pair pass through the other.*

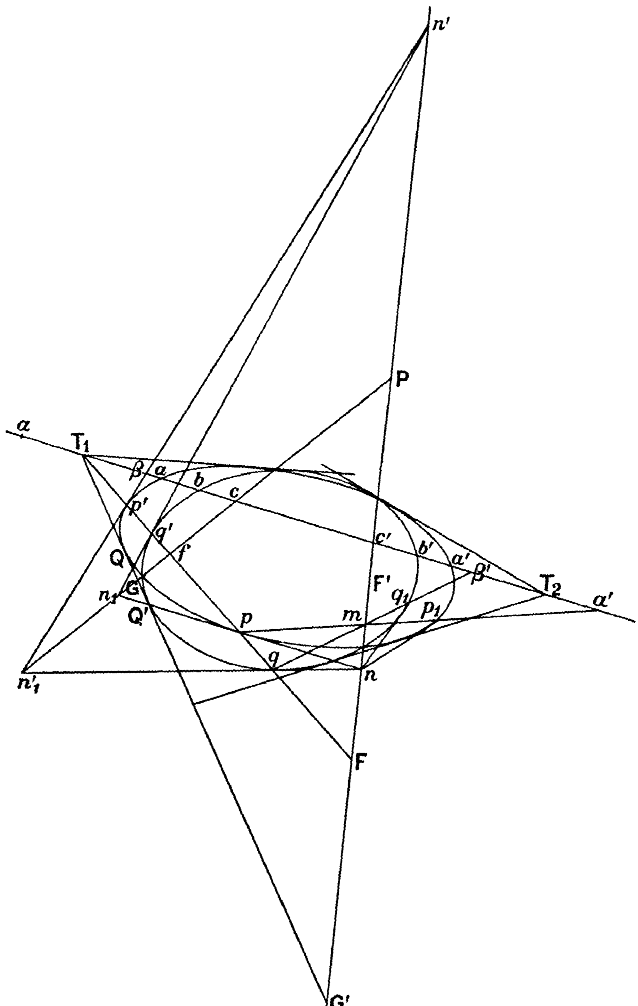

Fig. 108.

If through T_1 we draw any chord meeting the conic A in p and gg' in F, it will meet the conic B in two points q, q', and of these points one (q suppose) will be such that $(T_1Fpq) = \lambda'$, and the tangents at p, q will intersect in a point n which lies on gg', by Art. 244. Conversely, if on gg' we take a point n, and draw the four tangents np, nq, np_1, nq_1, the line joining the points of contact of two of these will pass through T_1. Let p, q be the pair. The line joining another pair of points of contact will also pass through T_1, for if we join T_1 to either of the points p_1 or q_1, it will obviously pass through the other, for the tangents at these points intersect in n.

We will now shew that the line joining the points of contact p, q_1 passes through T_2.

The lines pp_1 and qq_1 are the polars of n, and intersect in a point m which lies on gg', by Art. 210, and they pass respectively through the fixed points α', β', which are the poles of gg', and lie on T_1T_2, by Art. 227.

Since the triangle $m\alpha'\beta'$ is cut by the transversal T_1pq, by Menelaus' Theorem

$$\frac{mp}{\alpha'p} \cdot \frac{\alpha'T_1}{\beta'T_1} \cdot \frac{\beta'q}{mq} = 1 \quad \text{....................(1)}.$$

Since $(\alpha'p_1mp)$ is a harmonic range,

$$\frac{mp}{\alpha'p} = -\frac{mp_1}{\alpha'p_1}.$$

Since $(T_1T_2\alpha'\beta')$ is harmonic by Art. 229,

$$\frac{\alpha'T_1}{\beta'T_1} = -\frac{\alpha'T_2}{\beta'T_2}.$$

Therefore by substitution in (1) we obtain

$$\frac{mp_1}{\alpha'p_1} \cdot \frac{\alpha'T_2}{\beta'T_2} \cdot \frac{\beta'q}{mq} = 1.$$

Therefore, by the converse of Menelaus' Theorem, p_1q passes through T_2. Similarly q_1p passes through T_2.

We have thus proved that certain relations hold between a *pair* of tangent vertices and one of the corresponding common chords, and it could be shewn in exactly the same manner that the same relations hold for the same *pair* of tangent vertices and the other corresponding common chord, but of course the constant of homology would have a different value.

Hence, if we take any point on one of a pair of common chords, and from it draw the four tangents to the two conics, and take any pair of the points of contact which lie on different conics, then of the four lines which join them two will pass through one of the pair of tangent vertices corresponding to the common chords, and the other two will pass through the other tangent vertex.

247. *If from a tangent vertex we draw any transversal cutting the two conics in four real points, and draw the tangents at these points, the tangents to one conic will meet the tangents to the other in four points which lie by pairs on the two common chords corresponding to the tangent vertex.*

For let the transversal through T_1 cut the conics in p, q, and let the tangent to A at p meet the common chords in n, n_1. Then by Art. 246 of the two tangents from n to B one of the points of contact lies on T_1p and is therefore q, *i.e.* the tangent at p meets the tangent at q on the common chord gg'. Similarly it may be shewn that the other points of intersection of the tangents at p, q, p', q' lie on one or other of the pair of common chords corresponding to T_1.

In Fig. 108 the four points are n, n_1, n', n_1'.

248. From Art. 246 we see that in Fig. 108

If m is any point on a common chord Pg, poles α', β', and $m\alpha'$, $m\beta'$ meet the conics in two pairs of points p, p_1, q, q_1; the lines joining the pairs of these points which lie on separate conics will pass by pairs through the points T_1, T_2 corresponding to the common chord Pg.

Also $\lambda' = (T_1Fpq) = (T_1c'\alpha'\beta')$, by Art. 41 (1).

Relations between the four constants of homology.

249. In Fig. 108 let $T_1pp'qq'$ be any transversal through T_1. We have shewn in Art. 247 that the tangents at p, q intersect on the common chord PF, as do also the tangents at p', q', *i.e.* T_1 is a centre and PF the corresponding axis of homology of the two curves, the constant of homology being λ' suppose. Also by Art. 247 the tangents at p, q' intersect on the common chord Pf, as do also the tangents at p', q; *i.e.* T_1 is a centre and Pf the corresponding axis of homology of the two curves, the constant of homology being λ suppose.

Similarly T_2 is a centre of homology, and has Pf and PF as its corresponding axes according as we take for constants μ or μ'.

By Art. 248 these four constants of homology are determined by the equations

$$(T_1 c\alpha\beta) = \lambda, \qquad (T_1 c'\alpha'\beta') = \lambda',$$
$$(T_2 c\alpha\beta) = \mu, \qquad (T_2 c'\alpha'\beta') = \mu'.$$

Writing λ in the form $\dfrac{T_1\alpha}{T_1\beta} : \dfrac{c\alpha}{c\beta}$, &c. we see that these equations are connected by the relations

$$\frac{\lambda}{\mu} = \frac{T_1\alpha}{T_1\beta} : \frac{T_2\alpha}{T_2\beta} = (T_1T_2\alpha\beta) = -1, \text{ by Art. 229.}$$

Similarly $$\frac{\lambda'}{\mu'} = (T_1T_2\alpha'\beta') = -1.$$

Also, if $T_1QGQ'G'$ is a common tangent,

$$\lambda = (T_1GQQ'), \quad \lambda' = (T_1G'QQ').$$

$$\therefore \frac{\lambda}{\lambda'} = \frac{G'Q}{G'Q'} : \frac{GQ}{GQ'} = (G'GQQ') = -1, \text{ by Art. 213, Cor.}$$

$$\therefore \lambda = -\lambda' = -\mu = \mu'.$$

250. The results obtained in the previous article may be stated as follows:

Considering two conics as homologous figures, if we take one of a pair of tangent vertices as centre of homology, and one of the

pair of corresponding common chords as axis, the constant of homology has the same value both in magnitude and sign as when we take the second tangent vertex as centre of homology and the second corresponding common chord as axis; and it has its value the same in magnitude but of opposite sign when we take the first vertex as centre with the second chord as axis, or the second vertex as centre with the first chord as axis.

251. In Figs. 107, 108 let PG intersect A and B in the points g_1, g_4. Then retaining the same centre T_1, axis PG, and constant λ, if we construct the homologue of B we shall obtain a third conic C, which will touch the common tangents from T_1 to A and B, and pass through the points g_1, g_4; but the second centre and axis will not be T_2 and PG'. Similarly a fourth conic can be obtained from C, and so on. This system is a particular case of that considered in Art. 288.

If $\lambda = -1$, the homologue of B will be A, and the system will in this case consist of these two conics only.

Harmonic homology. For $(T_1GPP') = -1 = (T_1GP'P)$. This is called *harmonic homology* *.

Also since $(T_1c'\alpha'\beta') = 1$, T_1 coincides with c', and therefore the centre T_1 lies on the axis PG'. Similarly T_2 lies on the axis PG.

EXAMPLES.

1. Shew that if the conics described in Art. 251 are denoted by A, B_1, B_2, B_3... the successive conics B_1, B_2, B_3... are the homologues of A for the constants λ, λ^2, λ^3 ... respectively.

2. In the general case shew that the second centres for any pairs of the system are collinear, and that all the second axes are concurrent.

3. If with the same centre and axis, and constants λ, $-\lambda$ respectively, two conics B_1, B_2 are homologues of A, shew that B_1 and B_2 are in harmonic homology.

* Russell, *Elementary Treatise on Pure Geometry*, p. 325.

CHAPTER XVI

CONSTRUCTION OF COMMON CHORDS AND TANGENT VERTICES AND COMMON SELF-CONJUGATE TRIANGLE OF TWO CONICS

252. We will now consider the reality and imaginarity of the different groups each consisting of a *pair* of common chords and the corresponding *pair* of tangent vertices, and we will shew how to construct them when it is possible to do so.

253. Poncelet in his *Prop. Proj.* Vol. I, Art. 54 divides common chords into three classes:

(1) *real*, when they pass through two real points of intersection of the curves,

(2) *ideal*, when the points of intersection of the curves through which they pass are unreal, but the chords themselves are real,

(3) *imaginary*, when the chords cannot be constructed.

Similarly, tangent vertices may be

(1) *real*, when they are the intersection of two real common tangents,

(2) *ideal*, when the tangents through them are unreal, but the points themselves are real,

(3) *imaginary*, when the points cannot be constructed.

254. I. *When the conics intersect in four real separate points.*

In Fig. 102 the three groups

$$T_1,\ T_2,\ g_1g_4,\ g_2g_3 \ \dots\dots\dots\dots\dots\dots\dots (1),$$

$$T_3,\ T_4,\ g_1g_2,\ g_3g_4 \ \dots\dots\dots\dots\dots\dots\dots (2),$$

$$T_5,\ T_6,\ g_1g_3,\ g_2g_4 \ \dots\dots\dots\dots\dots\dots\dots (3),$$

are all real, as are also the three vertices P_1, P_2, P_3 of the common self-conjugate triangle.

255. II. *When two of the points, as g_1, g_4, coincide, whilst the other two, g_2, g_3, are real and separate.*

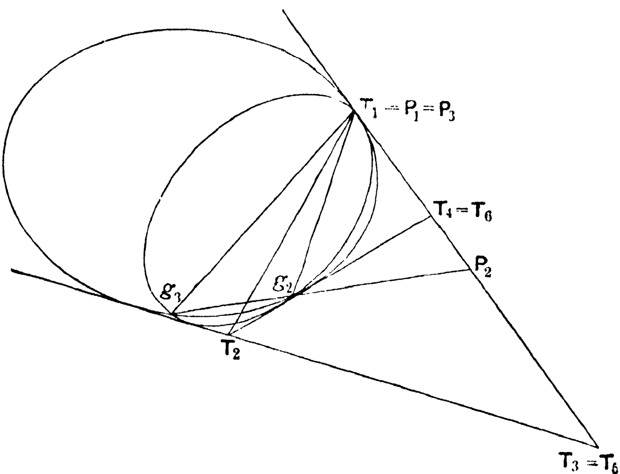

Fig. 109.

Here
$$g_1 = g_4 = T_1 = P_1 = P_3,$$
$$T_4 = T_6 \text{ and } T_3 = T_5.$$

The three groups are all real, but (2) coincides with (3).

256. III. *When the conics have double contact along the line LM, N being the intersection of tangents at L, M.*

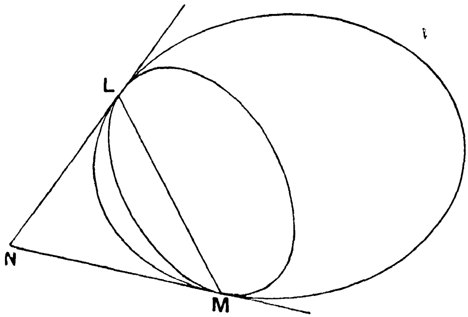

Fig. 110.

$$L = g_1 = g_4 = T_1 = P_1,$$
$$M = g_2 = g_3 = T_2 = P_3,$$
$$N = T_4 = T_3 = T_5 = T_6 = P_2.$$

P_1 and P_3 are indeterminate, being any pair of conjugate points on LM.

The three groups are all real, being

$$M, \ L, \ LN, \ MN,$$
$$N, \ N, \ LM, \ LM,$$
$$N, \ N, \ LM, \ LM,$$

i.e. (3) coincides with (2).

257. IV. *When the conics osculate at the point L the three groups are real and identical, being N, L, LM.*

$$L = g_1 = g_2 = g_4 = T_1 = T_4 = T_6 = P_1 = P_2 = P_3,$$
$$N = T_2 = T_3 = T_5.$$

258. V. *When the conics have four consecutive points common.*

The three groups are all real and identical, the common chords being the common tangent, and the tangent vertices all coinciding at the point of contact.

259. VI. *When the conics intersect in only two real points.*

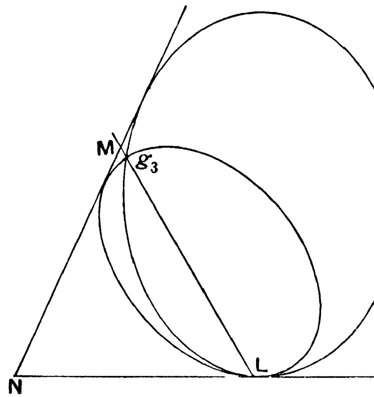

Fig. 111.

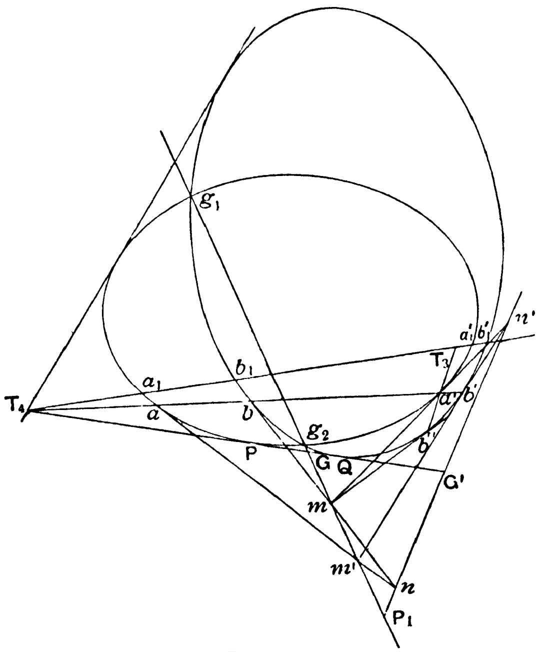

Fig. 112.

First method.

Let the two real common tangents intersect in T_4, through which draw a transversal meeting the conics in the four real points a, a', b, b'. Let the tangents at a, b meet in h, and let them intersect the real common chord g_1g_2 in m, m'. Then by Art. 247 ma', $m'b'$ are the tangents at a', b'; produce them to meet in n'.

By Art. 247 n, n' are two points on the corresponding common chord, which is ideal. Join nn', meeting g_1g_2 in P_1, which has one common polar for the two conics. Draw the tangent mb'' and also the common polar of P_1, which will pass through T_4 by Art. 226, and let it meet $b''a'$ in T_3. Then by Art. 246 T_3 is the real point of intersection of a pair of imaginary common tangents, and is an ideal tangent vertex.

Let the polar of P_1 meet the conics in a_1, a_1', b_1, b_1'. By Art. 230 P_1P_2, P_1P_3 are the double rays of the overlapping involution $P_1(a_1a_1', b_1b_1')$, and are consequently unreal.

Therefore the points P_2, P_3 are imaginary, and lie on the real line T_3T_4.

A little consideration will shew us that the other two *pairs* of common chords must be imaginary. For by Def. of Art. 209, two common chords which do not belong to the same *pair* must intersect at a point where the curves also intersect. Consequently, if it were possible to construct a second *pair* of common chords, they would meet the *pair* already drawn in points where the curves intersect, which is contrary to the hypothesis that the curves intersect in only two real points.

Second method.

Draw any transversal meeting the curves in α, α' and β, β', and the real common chord in γ, and find γ' such that the range $(\alpha\alpha', \beta\beta', \gamma\gamma')$ is in involution, Arts. 104, 108 ad fin. Then by Art. 213, γ' is a point on the second common chord.

Similarly, by drawing any other transversal we can find a second point γ'' on the second common chord. The line joining $\gamma'\gamma''$ is the line required.

Third method.

Let the common tangent T_4PQ meet the real common chord in G, and on it find the point G' such that the range $(PQGG') = -1$, Arts. 29, 32, Cor. Then, by Art. 213, Cor., G' is a point on the second common chord.

Similarly from the other common tangent we can find another point G'' on the second common chord.

260. VII. *When the two conics touch externally.*

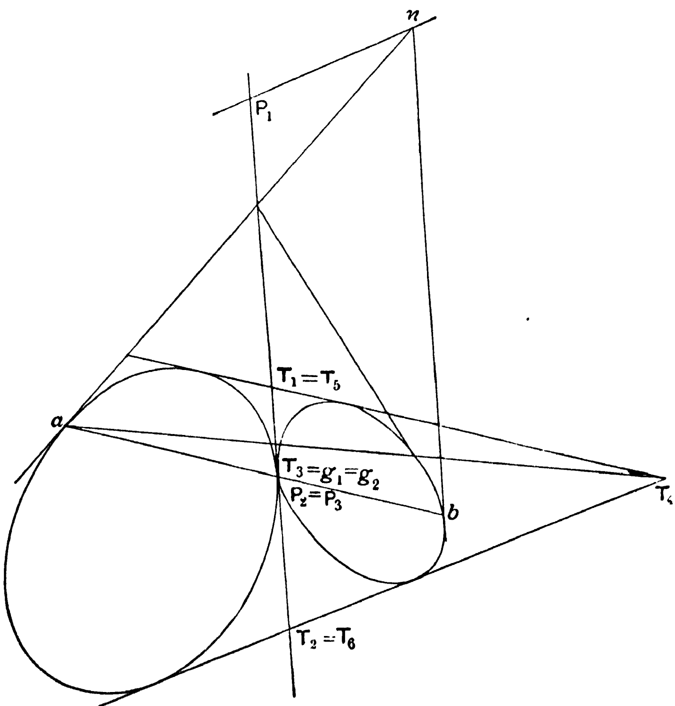

Fig. 113.

The common tangent at the point of contact T_3 is also a real common chord. The line T_3T_4 is a common polar by Art. 225, and its pole P_1 is the point of intersection of a *pair* of common chords. Through T_3 draw any transversal meeting the conics in a, b. If the tangents at a, b meet in n, the line P_1n is the second common chord (ideal) by Art. 247.

The three groups consist of

$$(2)\ T_3,\ T_4,\ P_1T_3,\ P_1n,$$

(1) and (3) T_1, T_2, and two imaginary lines intersecting in the real point T_3.

261. VIII. *When the conics touch internally.*

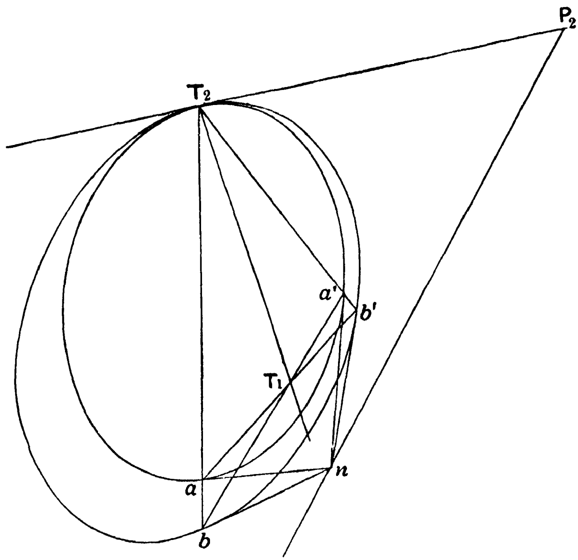

Fig. 114.

The tangent at T_2 is a real common chord. Through T_2 draw any transversal T_2ab meeting the conics in a, b. By Art. 247 the tangents at a, b meet at a point n on the second common chord.

Through n draw the other two tangents, touching the conics in a', b'. Let ab', $a'b$ meet in T_1 (an ideal tangent vertex). Let P_2 be the pole of T_1T_2. Then P_2n is the second common chord (ideal). P_1 and P_3 coincide at T'_2.

262. IX. *When the conics have no real points of intersection, each conic being entirely without the other.*

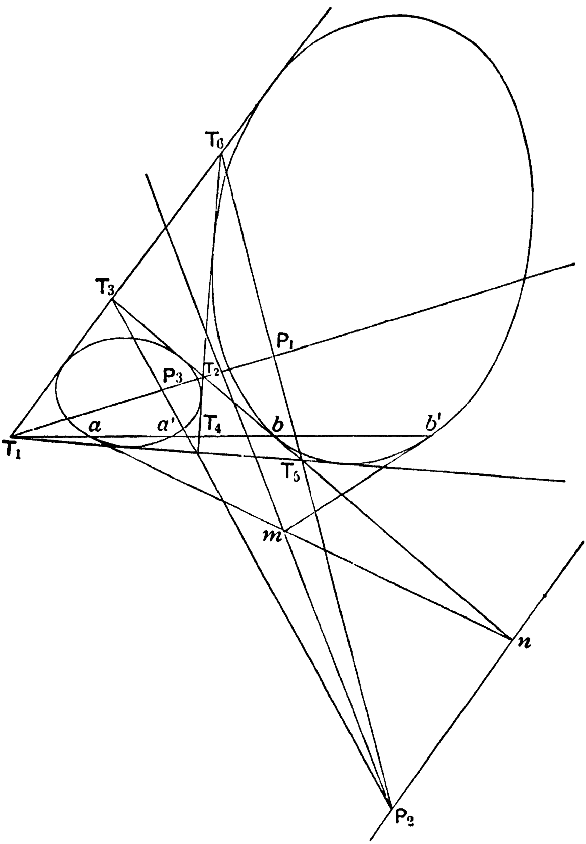

Fig. 115.

Let a transversal through T_1 cut the conics in aa', bb'. Let the tangents at a, b' meet in m, and those at a, b in n, and find P_2 the pole of T_1T_2 with respect to either conic. Then by Art. 247 P_2m, P_2n are a *pair* of ideal common chords. The other common chords are imaginary, one *pair* intersecting in the real point P_1, the other *pair* in the real point P_3.

263. X. *When the conics have no real points of intersection, one being entirely within the other.*

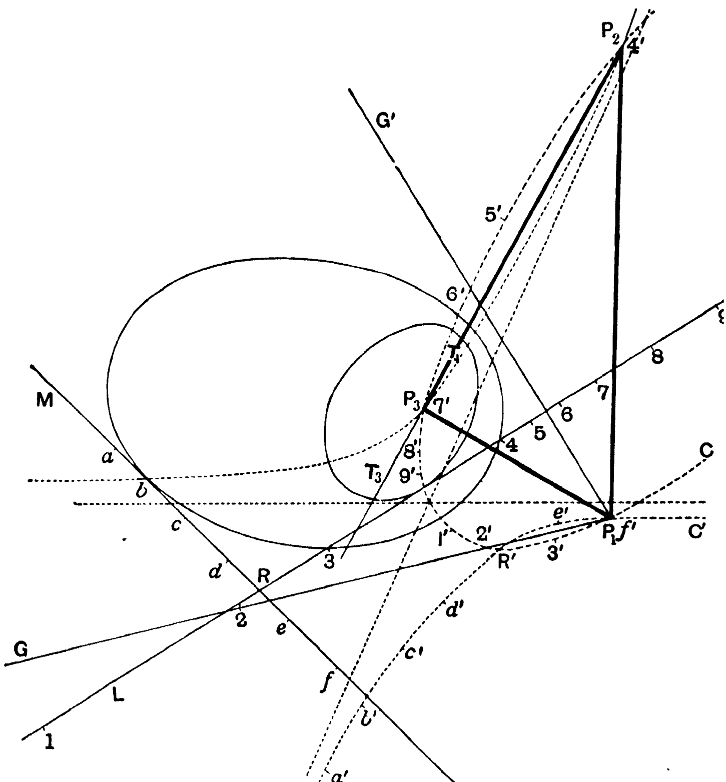

Fig. 116.

To solve the problem in this case we will make use of the property of Art. 219, viz. if we have a straight line L, and if P' be the intersection of the polars of any point P on L, then as P moves along L, P' describes the eleven-point conic passing through the poles of L for A and B.

Denote the reciprocal conic by C, and describe it by points, denoting the different points on L by the numbers 1, 2, 3...n..., and the points on C corresponding to them by 1′, 2′, 3′...n'....

Take another line M, and construct its reciprocal conic C', denoting the different points on M by the letters a, b, c...m..., and the reciprocal points on C' by a', b', c'...m'....

Then it is clear that if L and M intersect in R, its reciprocal point R' will be common to C and C'.

Let P_1, P_2, P_3 be the other three points of intersection of C and C'. These will always be real, except in Case VI, Art. 259, where A and B have only two real points of intersection, when only one of the points P_1, P_2, P_3 will be real.

Suppose we consider one of the points as P_1. Regarded as a point on C, the two polars of P_1 intersect at a point on L, and as a point on C' they intersect at a point on M. Consequently they must coincide, and hence P_1 is a point which has the same polar for A and B, and similarly for P_2 and P_3. Therefore, by Art. 212, $P_1P_2P_3$ is the common self-conjugate triangle of A and B.

If n, n' denote any pair of reciprocal points, they are conjugate points, by Art. 219, the polar of n for the pair of common chords through P_1 (say), which is one of the conics of the system, passes through n', and the rays P_1n, P_1n' form a harmonic pencil with the common chords through P_1, Art. 121, Def.

The same property holds for all pairs of reciprocal points (n, n'). Also P_2 and P_3 are conjugate points. Therefore, giving to n the different values 1, 2, 3..., by Art. 111,

$$P_1(11', 22', 33' \ldots nn' \ldots P_2P_3)$$

forms a pencil in involution in which the double rays are the

common chords through P_1, and similarly for the common chords through P_2 and P_3.

Now the pair of common chords through P_3 is clearly imaginary. And if we give to n any value such as 5 or 8, we see from the figure that the pencil $P_1(P_2P_3, nn')$ is non-overlapping, whilst $P_2(P_1P_3, nn')$ is overlapping. Consequently the common chords through P_1 are real, and those through P_2 are imaginary.

It is obvious from the figure that one of the double rays through P_1, viz. P_1G, cuts L between the points 1 and 2, near 2, and the other, P_1G', cuts it between 5 and 6, at about 5·7. Hence P_1G and P_1G' are a *pair* of ideal common chords.

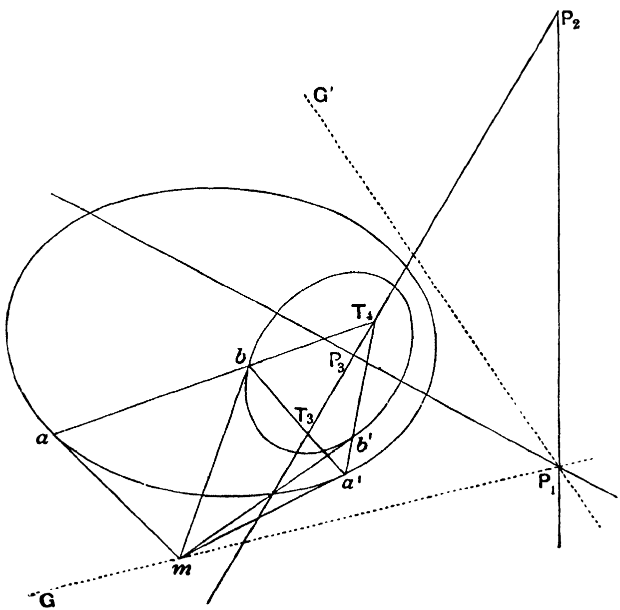

Fig. 117.

To find the tangent vertices take any point m on one of the common chords P_1G, draw the tangents ma, ma', mb, mb'. Then, by Art. 246, $a'b$, ab will intersect P_2P_3 in the points T_3, T_4, a *pair* of ideal tangent vertices.

CHAPTER XVII

CONICS HAVING DOUBLE CONTACT

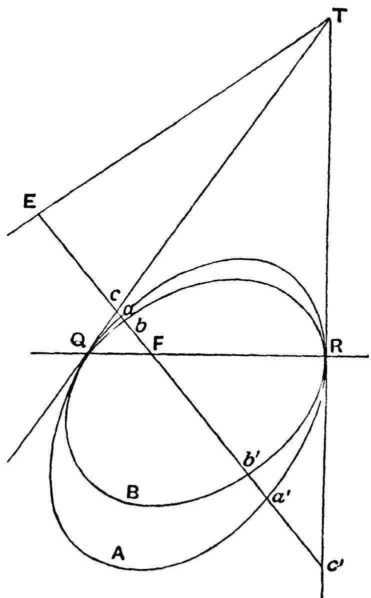

Fig. 118.

264. If in Fig. 102 the points g_1 and g_2 move up to and coincide with one another, as also the points g_3 and g_4, the two conics will touch one another at the points Q, R, as shewn in Fig. 118, and are said to have *double contact* with each other.

Conics having Double Contact.

The *pair* of common chords g_1g_4, g_2g_3 coincide in the line QR, which is called the *chord of contact*, and the corresponding *pair* of tangent vertices T_1, T_2 coincide in the point T, which is called the *pole of contact*.

Chord of Contact. Pole of Contact.

The *pair* of common chords g_1g_3 and g_2g_4 will also coincide with QR, and their corresponding *pair* of tangent vertices T_5, T_6 coincide with T.

Also the tangents TQ, TR are the limiting positions of the *pair* of common chords g_1g_2, g_3g_4, of which the corresponding *pair* of tangent vertices T_4, T_3 are at Q, R respectively.

265. *Any two conjugate points on the chord of contact form with the pole of contact a common self-conjugate triangle.*

For if F, G are the two conjugate points, FT is the polar of G, GT is the polar of F, and T is the pole of GF. Hence

Any point on the chord of contact has the same polar for both conics, and this polar passes through the pole of contact.

In other words:

If any transversal is drawn through the pole of contact, the tangents at the points where it cuts the conics all pass through the same point on the chord of contact.

This also follows from Art. 247.

266. If any transversal meets the conics in aa', and bb', and the chord of contact in F, the point F, by Art. 187, is obviously a double point of the involution range whose characteristic is (aa', bb'). Hence

Any tangent to the one is cut harmonically at its point of contact, and at the points where it meets the chord of contact and the other conic.

267. *The polars of any point E intersect on the chord of contact.*

Let the polar of E for the conic A meet QR in F, and let EF meet the conics A and B in aa', and bb'. Then by Art. 187 F is one double point of the involution range given by (aa', bb'), and E, its harmonic conjugate for a, a', is obviously the other. Hence the polar of E for B passes through F.

COR. *Of two conjugate points one is always on the chord of contact.*

268. Again, if from any point E we draw pairs of tangents to A and B, the line ET is a double ray of the involution pencil determined by these pairs of tangents by Art. 228, since a *pair* of tangent vertices coincide at T.

269. *The poles of any straight line L are collinear with the pole of contact.*

Let l_A, l_B be the poles, and let the line joining them meet L in P, and from P draw the pairs of tangents to A and B. Then by Art. 233 Pl_Al_B is a double ray of the involution pencil determined by these tangents, and therefore, by Art. 188, passes through T.

The two lines L and l_Al_B are conjugate, hence

Of two conjugate lines one always passes through the pole of contact.

270. *If a common chord of two conics has the same pole, the conics have double contact along the chord, and if a tangent vertex of two conics has the same polar, they have double contact along the polar.*

271. *If two conics have double contact, and through the points of contact a third conic is drawn, its corresponding common chords with each of the conics will intersect on the chord of contact.* Art. 215.

272. *If two conics A and B have each double contact with a third conic C, a pair of their common chords pass through the point of intersection of the two chords of contact, and form a harmonic pencil with them*.*

For the intersection P of the chords of contact is the pole of the line joining the poles of contact T, T', and therefore has the same polar for all three conics; consequently, by Art. 212, it is the intersection of a *pair* of common chords of A and B.

PT, PT' form with the pair of common chords through P a harmonic pencil by Art. 230.

273. *If any transversal through the pole of contact T meets the conics in aa′, bb′, then as the transversal rotates about T the value of the cross-ratio (Tbaa′) remains constant, and is the reciprocal of (Tb′aa′).*

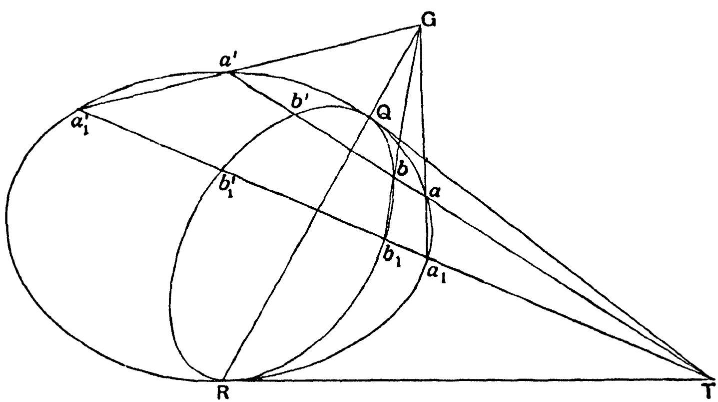

Fig. 119.

Let $Ta_1b_1b_1'a_1'$ be any other position of the transversal.

Then, by Art. 244, aa_1, bb_1 meet on the chord of contact, as do also aa_1, $a'a_1'$, by Art. 161. Therefore the three chords

* Poncelet, *Prop. Proj.* Art. 427; Chasles, *Sect. Con.* Art. 415; Salmon, Art. 263.

aa_1, $a'a_1'$, bb_1 all pass through the same point G on the chord of contact, and by Art. 161 $b'b_1'$ passes through the same point. Therefore the ranges $(Tbaa')$ and $(Tb_1a_1a_1')$ have T for a corresponding common point, and are in perspective, centre G, and their cross-ratios are consequently equal by Art. 21. And since these are any two positions of the transversal, the cross-ratio $(Tbaa')$ is the same for all positions of it.

Again, by Art. 230 T is one of the double points of the involution determined by (aa', bb').

Therefore $(Tbaa') = (Tb'a'a) = \dfrac{1}{(Tb'aa')}$ by Art. 3.

Conversely we have:

If through a given point T a transversal is drawn meeting a conic in aa', and on it a point b is taken such that the cross-ratio $(Tbaa')$ is constant, the locus of b is a conic having double contact with the given conic along the polar of T.

274. *The chords which join pairs of corresponding points of two homographic rows on a conic envelop a second conic which has double contact with the given one at the common points of the rows.*

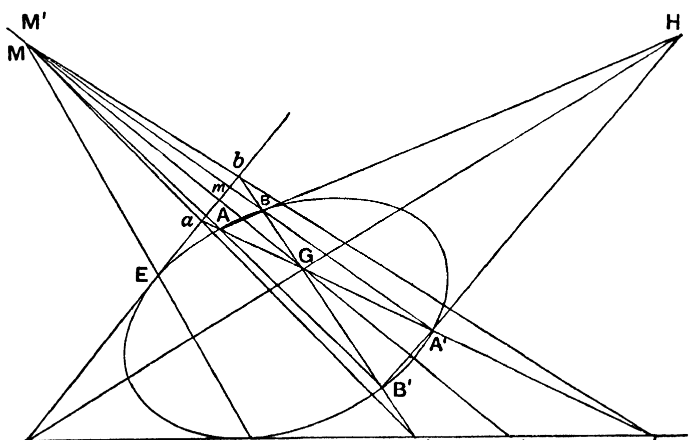

Fig. 120.

Let AA', BB' be pairs of corresponding points of two homographic rows on the conic. Let E, F be the common points of the rows. Then by Art. 157 EF is the cross-axis on which pairs of chords joining pairs of corresponding points taken inversely intersect. Let AB', $A'B$ intersect on EF in M, and produce AA', BB'... to meet the tangents at E, F in aa', bb'.... We will shew that these are pairs of corresponding points of two homographic ranges.

Consider the inscribed quadrangle $ABA'B'$. Its diagonal points are G, H, M. Therefore HG is the polar of M for the conic. But since M is on EF, its polar passes through T. Therefore TGH is a straight line, and the following are harmonic:

$$G(TMAB),\quad (Tmab),\quad (Tm'a'b'),\quad (Tm'b'a').$$

Therefore $(Tmab) = (Tm'b'a')$, and ab', ba' intersect on mm', *i.e.* on MG.

Let them intersect in M'. Join TM'.

Then in the quadrangle $aba'b'$, T, G, M' are the diagonal points. Therefore TG is the fourth harmonic of TM' for TE, TF. But from the conic, since TG is the polar of M, TG is the fourth harmonic of TM for the tangents TE, TF. Hence TM' coincides with TM, and M' with M. Therefore ab', $a'b$ intersect in M, *i e.* on the fixed line EF, which is consequently the cross-axis of the two homographic ranges of which aa', bb'... are pairs of corresponding points. Therefore, by Art. 139, aa', bb'... are tangents to a conic which touches TE, TF at the points E, F. Conversely,

If two conics C and C' have double contact at E, F, and a chord AA' of C rolls upon C', its extremities A, A' form two homographic conic-pencils whose common points are at E, F.

COR. 1. *If TP, TQ are two fixed tangents to a conic, R and S two variable points on the curve such that either* (1) $(PQRS)$ *or* (2) $T(PQRS)$ *is constant, the chord RS will envelop a conic having double contact with the given conic at P, Q, and conversely.*

For by Arts. 159, 192 (R) and (S) form two homographic divisions on the conic, having P, Q for common points.

Cor. 2. *If tangents are drawn at pairs of corresponding points of two homographic divisions on a conic, the locus of their point of intersection is a conic having double contact with the former at the common points of the rows.*

This theorem, which is the correlative of Art. 274, can easily be proved by the method of Art. 275.

275. *If the locus of a point* α *is a conic* C, *the envelope of its polar for a conic* C' *is a conic* C''; *and conversely, if a straight line* aa' *moves so as to envelop a conic* C'', *the locus of its pole for a conic* C' *is a conic* C*.

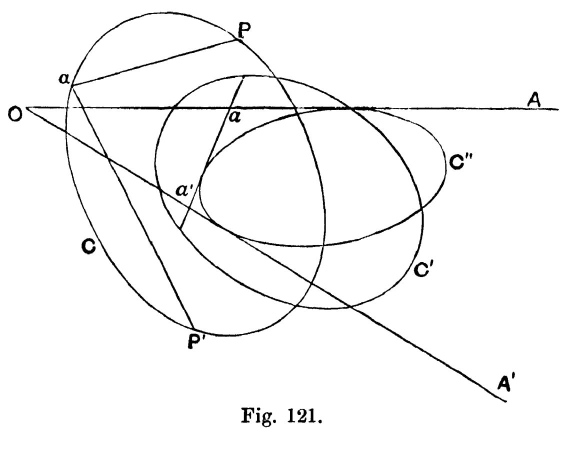

Fig. 121.

Let P, P' be two fixed points on C, and let α, β, γ... be any other points on C. Let OA, OA' be the polars of P, P' for C', and let the polars of α, β, γ... for C' meet OA in a, b, c... and OA' in a', b', c'....

* Poncelet, *Prop. Proj.* Art. 231.

Then Pa is the polar of a, and $P'a$ the polar of a', &c., and since by Art. 167

the range of poles $(abc\ldots)$ = the pencil of polars $P(\alpha\beta\gamma\ldots)$,

and

the range of poles $(a'b'c'\ldots)$ = the pencil of polars $P'(\alpha\beta\gamma\ldots)$,

and by Art. 129 $\quad P(\alpha\beta\gamma\ldots) = P'(\alpha\beta\gamma\ldots)$,

therefore $\quad (abc\ldots) = (a'b'c'\ldots)$,

and by Art. 139 the lines aa', $bb'\ldots$ envelop a conic touching OA and OA'.

To prove the converse, let OA, OA' be two positions of the moving line aa', and let P, P' be the poles of OA, OA' for C'.

Then by Art. 130 $\quad (abc\ldots) = (a'b'c'\ldots)$,

therefore by Art. 167 $\quad P(\alpha\beta\gamma\ldots) = P'(\alpha\beta\gamma\ldots)$,

and by Art. 138 the locus of α is a conic through P and P'.

276. *If a triangle ABC inscribed in a conic moves so that two of its sides pass through fixed points P, P', its third side will envelop a conic having double contact with the given conic at the points where the latter is met by the line PP'* *.

As the triangle moves, the conic pencils (A) and (B) being homographic to (C) are homographic to each other by Art. 186. Therefore by Art. 274 AB envelops a conic having double contact with the given conic at e and f.

277. *If a triangle circumscribing a conic moves so that its base angles move along fixed straight lines OM, OM', its vertex will describe a conic having double contact with the given conic at the points where the latter is met by the polar of O* †.

In Fig. 122 through A, B, C draw the tangents forming the circumscribing triangle abc, and let O be the pole of PP'. Then

* Poncelet, *Prop. Proj.* Art. 431.

† Poncelet, Art. 435; Salmon, Art. 272, Exs. 2, 3.

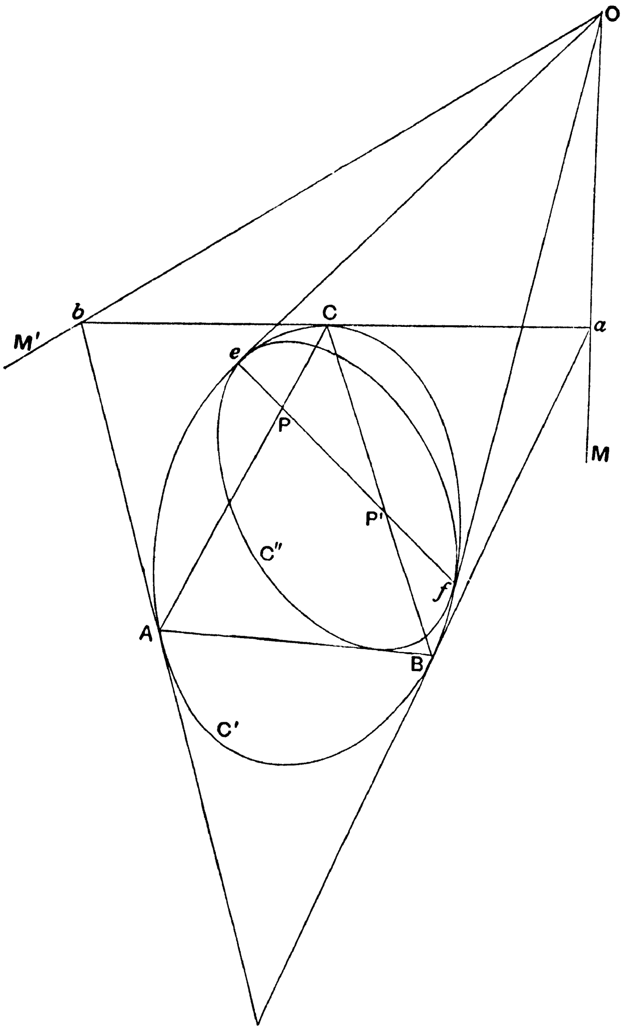

Fig. 122.

as the triangle ABC moves it is evident that the points a, b, being the poles of BC, CA, will move along the fixed lines OM, OM' the polars of P, P', and by Art. 275 the point c will describe a conic. Then use Art. 274, Cor. 2.

EXAMPLES.

1. A, B are two fixed points on a conic, and L is a given straight line meeting it in M, N. P is a variable point on L, and AP, BQ meet the conic again in Q, R. Shew that the envelope of QR is a conic having double contact with the given conic at the points M, N, and touching the line AB.

2. CA, CB are two given tangents to a conic, and D a fixed point in the plane. Any transversal through D meets CA, CB at E, F, and from E, F other tangents are drawn meeting in T. Shew that the locus of T is a conic passing through C, and having double contact with the given conic at the points of contact of the tangents from D.

3. Two conics A, B have double contact at Q, R; T being the pole.

(1) If P is a variable point on A, and PQ, PR meet B in D, E, then DE envelops a conic having double contact with A and B at Q and R.

(2) If FG is any chord of A which is also a tangent to B, and if QF, RG intersect in H, the locus of H is a conic having double contact with A and B at Q and R.

4. If two conics have double contact the cross-ratio of four of the points in which any four tangents to the one meet the other is the same as that of the other four points in which the four tangents meet the curve, and also the same as that of the four points of contact. [Townsend.]

5. aa', bb', cc' are three fixed chords of a conic. Shew that the envelope of a fourth chord dd' such that $(abcd) = (a'b'c'd')$ is a conic having double contact with the given conic.

6. The locus of the intersection of tangents to a conic which divide a finite segment II' of a given tangent in a constant cross-ratio is a conic having double contact with the given conic at the points of contact of tangents from I and I'.

7. Two conics A and B have double contact at Q, R. If through two points m, m' on B we draw tangents to it meeting A in the two pairs of points a, b and a', b', then the two chords aa', bb' will pass through the intersection of the lines QR and mm'.

8. In Ex. 7 if from two points n, n' on A we draw chords touching B and forming the circumscribed quadrilateral $abcd$, one diagonal of this quadrilateral will pass through T the pole of contact of the conics, and through the pole of nn' for A.

9. In Ex. 7, if the vertex p of an angle circumscribing B moves along A, the points m, m' where the sides of the angle meet A form two homographic divisions which have Q, R for common points, and the chord mm' envelops a conic having double contact with the given conics at Q, R.

CHAPTER XVIII

CONSTRUCTION OF A CONIC SATISFYING CERTAIN CONDITIONS

278. *To describe a conic through five given points.*

This problem has been fully solved in Art. 140 by what we may term the *first method*, due to Chasles. On account of the importance of the question, and the frequent reference that is made to it in constructions connected with the conic, we have given a few other methods so that when a student is told to "describe a conic through five points" he may select any one of the methods and know exactly what the words imply.

Second method.

Let a, b, c, O, O' be the given points. By Art. 138 the locus of the intersections of corresponding rays of two homographic pencils not in perspective is a conic passing through the centres of the pencils. If then we take two of the five points O, O' as centres, and join each of them to the remaining three points a, b, c we shall obtain two pencils each containing three rays, and if through one of the centres, O, we draw any fourth ray Od, and construct the ray corresponding to it in the second pencil, the point δ where these two rays intersect will be a point on the conic. This problem is solved completely in Art. 48, and by drawing different rays through O, and repeating the construction, we can obtain as many points δ on the curve as we please.

Third method, employing Pascal's Theorem, Art. 146.

Through γ the intersection of Ob, $O'a$ draw any straight line

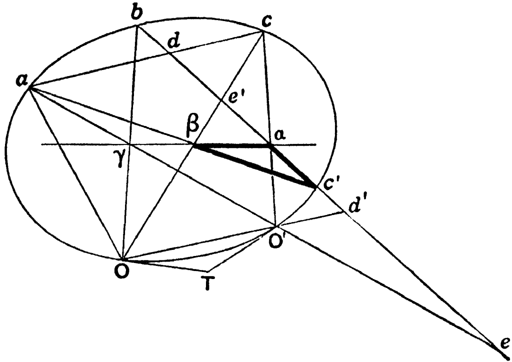

Fig. 123.

meeting $O'c$ in α and Oc in β. Join $\alpha\beta$ and $b\alpha$ meeting in c'. Then by Art. 147 c' is a point on the curve, and by drawing different lines through γ, and treating them as Pascal lines, we can obtain other points on the curve.

Fourth method, employing Maclaurin's Theorem, Art. 151.

In Fig. 123 let aO', Ob meet in γ. Through a draw any straight line ac' meeting Oc in β, and join $\beta\gamma$ meeting $O'c$ in α. Then c', the intersection of $a\beta$ and $b\alpha$, is a point on the curve, for it is the vertex of the triangle $c'\alpha\beta$ whose base angles β and α move along the fixed lines cO, cO', and whose sides pass through the fixed points a, b, γ.

Fifth method, by Desargues' Theorem, Art. 187.

Consider $OacO'$ as an inscribed quadrangle, and through b draw any transversal meeting the opposite sides ac, OO' in d, d', and the diagonals $O'a$, Oc in e, e', and on the transversal find the point c' such that (dd', ee', bc') is an involution range, Arts. 104, 108 *ad fin.* Then by Art. 187 c' is a point on the required conic.

Other methods might be given, but these are sufficient to shew the application of the theory of cross-ratio to the problem.

279. *To draw the tangent at O', any one of the five given points.*

If we take a point d on the curve very near to O', since Od, $O'd$ are corresponding rays in the two homographic pencils centres O and O', it is obvious that the tangent at O' is the ray in the pencil whose centre is O' corresponding to the ray OO' in the pencil centre O; and consequently the rays constructed in Art. 58 are the tangents at O and O', their intersection T being the cross-centre of the pencils.

280. *To find the points where the conic through five points meets a given straight line L.*

Let the pencils $O(abc)$ and $O'(abc)$ meet L in the points α, β, γ and α', β', γ'. Then the points required are obviously the common points of the two homographic co-axial ranges of which $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$ are the characteristics. This problem is solved in Arts. 83—86.

281. *To find the directions of the asymptotes of the conic through five points.*

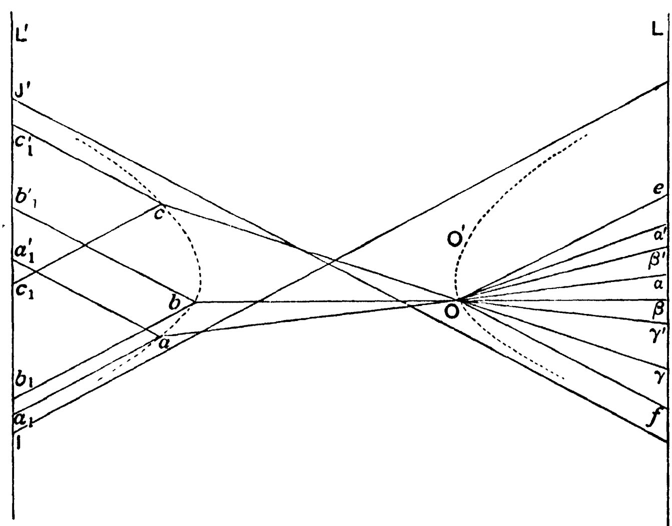

Fig. 124.

This is equivalent to the problem of finding a pair of parallel corresponding rays in the two pencils centres O, O'.

Through O draw three rays $O\alpha'$, $O\beta'$, $O\gamma'$ parallel respectively to $O'a$, $O'b$, $O'c$. Then we have two concentric pencils whose characteristics are $O\,(abc)$ and $O\,(\alpha'\beta'\gamma')$, and if we cut them by any transversal L, the common rays Oe, Of, which can be found by Art. 84, give us the required directions.

282. *To draw the asymptotes of the conic through five points.*

In Fig. 124, through a, b, c draw pairs of lines respectively parallel to Oe and Of (Art. 281), meeting any transversal L' in $a_1b_1c_1$ and $a_1'b_1'c_1'$. The parallel lines aa_1, bb_1, cc_1 are three rays of a pencil whose centre is at infinity along Oe, and aa_1', bb_1', cc_1' are three rays of a pencil whose centre is at infinity along Of. Consider the two ranges whose characteristics are $a_1b_1c_1$ and $a_1'b_1'c_1'$. Find by Art. 82 I the point on the first range corresponding to the point at infinity on the second, and J' the point on the second corresponding to the point at infinity on the first. Then the line through I parallel to Oe is one asymptote, and the line through J' parallel to Of is the other.

283. *To construct a conic given five tangents.*

In Fig. 79 let the five tangents be ab, aa', $a'b'$, bb', cc'.

Let ab, $a'b'$ meet in O. Consider the ranges on Oa, Oa' whose characteristics are abc and $a'b'c'$. Find d, d' any pair of corresponding points by Art. 40. Then by Art. 139 dd' is a tangent.

Similarly any number of tangents can be drawn, and the conic constructed by means of them.

284. *Given five tangents, to find the point of contact of any one of them, $a'b'$ suppose.*

First method.

Find by Art. 40 the point A' in the range $a'b'c'$ corresponding

to O in the range abc. By Art. 130 *ad fin.* A' is the point required.

Second method.

In Fig. 79 join ac' meeting $a'c$ in δ. Then $b\delta$ will meet $a'b'$ in the point required. For if d' moves up to and coincides with A', γ will coincide with c', etc.

285. *Given five tangents, to draw a pair of tangents from a given point P.*

In Fig. 79 let the given tangents be ab, aa', $a'b'$, bb', cc'. Join P to the points a, b, c and a', b', c'. Then the required tangents are evidently the common rays of the pencils whose characteristics are $P(abc)$ and $P(a'b'c')$, and can be constructed as in Art. 84.

286. *To construct a conic given four points and a tangent.*

Let a, b, c, d be the four points, L the given tangent. Consider L as a transversal meeting the opposite sides of the quadrangle $abcd$ in the pairs of points $\alpha\alpha'$, $\beta\beta'$. Then by Desargues' Theorem, Art. 187, e, f the double points of the involution determined by $(\alpha\alpha', \beta\beta')$ are the points of contact of L with the two conics which satisfy the conditions of the problem. These can then be constructed by one of the methods of Art. 278.

287. *To construct a conic given four tangents and a point.*

Let P be the given point, and join P to the pairs of opposite vertices of the quadrilateral formed by the four given tangents. Then by Art. 188 these four rays determine a pencil in involution in which the double rays are tangents to the two conics, which can then be constructed from the two sets of five tangents by Art. 283.

288. *If a system of conics is described passing through two given points a, b, and touching two given straight lines OT, OT′, (1) the polars of the point O pass through one or other of two fixed*

points on the line ab, and (2) *the poles of the line ab lie on one or other of two fixed lines passing through O.*

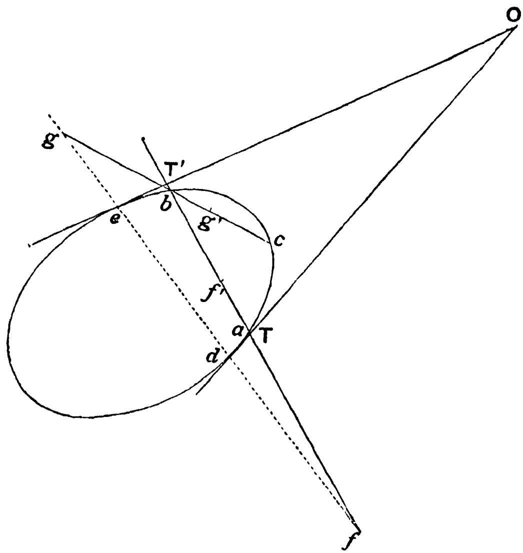

Fig. 125.

Let d, e be the points where any conic of the system touches OT, OT', so that de is the polar of O for that conic. Then the segment de may be considered as a quadrilateral inscribed in the conic, two of its opposite sides being the tangents at d and e, the other pair of opposite sides being represented by de considered as two coincident straight lines.

(1) Let ab meet OT, OT' in the points T, T', and de in f. Then by Desargues' Theorem, Art. 187, the two pairs of points (ab), (TT') determine a system in involution in which f is one of the double points; and if f' is the harmonic conjugate of f for a and b, f' is the other double point of the involution, by Art. 99. Hence the polars of O for the different conics of

the system pass through one or other of the known points f, f'.

(2) Again, considering the same conic we see that Of, Of' are harmonic conjugates both for (Oa, Ob) and for (OT, OT'), and therefore by Art. 161 the pole of ab lies on one of the lines Of or Of'. Hence the poles of ab for the different conics of the system lie on one or other of the known lines Of, Of'.

289. *To construct a conic given three points and two tangents.*

Let a, b, c be the three points, OT, OT' the two tangents.

It was shewn in Art. 288 that for the system of conics passing through the points a, b and touching the lines OT, OT', the polars of the point O pass through one or other of two known points f, f' on the line ab, see Fig. 125. Similarly for the system of conics passing through the points b, c and touching the same pair of lines OT, OT', the polars of the point O pass through one or other of two known points g, g' on the line bc.

Now the conics which pass through the three points a, b, c, and touch the lines OT, OT', are those which are common to the above two systems, and are therefore such that the polars of O pass through one of the points f, f', and also through one of the points g, g'. Hence there are four, and only four, polars, and consequently four, and only four, conics. If one of the polars meets OT, OT' in d, e, the problem is reduced to the construction of a conic through five points, Art. 278.

290. *To construct a conic given three tangents and two points.*

Let the points be a, b and the tangents OT, OT', TT', and let the line ab meet them in the points t, t', t'' respectively.

It was shewn in Art. 288 that for the system of conics passing through a, b and touching the lines OT, OT' the poles of the line ab lie on one or other of two known lines passing through O, viz. Of, Of', where f, f' are the double points of the

involution determined by (ab, tt'). Similarly for the system of conics passing through a, b and touching the lines OT', TT', the poles of the line ab lie on one or other of two known lines passing through T, viz. Tf_1, Tf_1', where f_1, f_1' are the double points of the involution determined by (ab, tt'').

Now the conics which pass through a, b and touch the three straight lines OT, OT', TT' are those which are common to the above two systems, and are therefore such that the poles of ab lie on one of the lines Of, Of', and also on one of the lines Tf_1, Tf_1'. Therefore there are four, and only four, poles, and consequently four, and only four, conics.

If one of the poles is P, then Pa and Pb are tangents, and the problem is reduced to the construction of a conic touching five lines, Art. 283.

CHAPTER XIX

HOMOGRAPHIC GENERALISATION OF CIRCLES AND THE CIRCULAR POINTS AT INFINITY, CONICS AND THEIR FOCI, AND OTHER ASSOCIATED POINTS AND LINES

291. WE touched briefly on the relations of the circular points with points and lines in Arts. 113, 114, and with circles and conics in Arts. 179—183. We will now consider them more fully. We shall use small letters to denote points in the original, and their capitals to denote the corresponding points in the generalised or derived figure. As the equations to the isotropic lines joining the origin to the circular points are $y = \pm ix$, we shall always denote these points in the original figure by the letters i, i', and their generalised positions by their capitals, I, I'. Since every circle passes through i, i', the points I, I' will only lie on a conic when it is generalised from a circle in the original figure. If a conic generalises into another conic so that the focus of the first becomes the intersection of two tangents to the second, the points I, I' are finite points on these tangents but not on the curve, being the points of contact only when the original conic is a circle, its centre, the pole of the line at infinity, becoming the intersection of the tangents, *i.e.* the pole of II'.

We will first give a list of the more important fundamental results which are obtained from the consideration of the circular points. These the student will easily verify from what we have said on the subject in the above quoted articles, and we will then shew how these results can be applied to obtain generalised properties of conics from the known properties of circles.

Data.

1. The line at infinity becomes a finite line, on which are two finite points I, I' corresponding to i, i' the circular points at infinity.

2. If c is the mid-point of a linear segment ab, and d the point at infinity on the line, and if A, B, C, D are the generalised positions of a, b, c, d, then D lies on II', and $(ABCD)$ is harmonic.

3. If c divides ab in a given ratio, with the notation of 2,

$$(ABCD) = (abc\infty) = \frac{ac}{bc}.$$

4. Lines which are parallel in the original figure become lines intersecting on II'.

5. Pairs of concurrent lines at right angles become pairs of lines which cut the segment II' harmonically.

6. Pairs of concurrent lines containing a constant angle become pairs of lines which cut the segment II' in a constant cross-ratio.

7. Pairs of concurrent lines containing angles bisected by a single pair of lines become pairs of concurrent lines cutting the segment II' in a series of points in involution in which I, I' are conjugate points, and the bisectors cut II' in the double points of the involution.

8. A circle becomes a conic through the points I, I', and the centre of the circle becomes the pole of II'.

9. A figure consisting of a conic, a pole and its polar can represent a circle, its centre, and the line at infinity.

10. A circle on ab as diameter becomes a conic through A, B, I, I', and having AB and II' for a pair of conjugate chords.

Since only one circle can be described on a given finite straight line as diameter, it follows that if we have given two

pairs of points in a plane, only one conic can be described passing through them, and having the line joining one pair conjugate to the line joining the other pair.

11. A focus is equivalent to the intersection of two tangents passing through the points I, I' which are not on the conic. The other focus becomes the intersection of the other tangents from I, I'.

12. A straight line through a given focus becomes a straight line through the intersection of two given tangents.

13. The tangents from the foci intersect in i, i', and therefore to have given two foci is equivalent to having given a quadrilateral circumscribing a conic, I, I' being a pair of opposite vertices.

14. Confocals become conics inscribed in a quadrilateral having I, I' for a pair of opposite vertices.

15. A parabola touches the line at infinity, and S being the focus, it has Si, Si' for tangents, and therefore Sii' is a tangent triangle; hence a parabola and its focus become a conic inscribed in a given triangle.

16. A rectangular hyperbola, having its asymptotes at right angles, has i, i' for conjugate points, and therefore becomes a conic cutting the segment II' harmonically.

17. Concentric circles become conics having double contact with one another at I, I'.

18. Conics having the same focus and directrix become conics having double contact.

19. Similar conics, having the angles between their asymptotes constant, become conics cutting the segment II' in a constant cross-ratio.

20. Co-axial circles become conics circumscribing the same quadrangle, two of whose vertices are I, I', the other two

vertices being the points A, B, corresponding to the points where the radical axis meets the circles.

In Fig. 102 let $g_1 = A$, $g_2 = B$, $g_3 = I$, $g_4 = I'$.

The line of centres becomes the line containing the poles of II', and is therefore the line P_2P_3, one of the sides of the common self-conjugate triangle, and p is the intersection of the radical axis and the line of centres. Since these two lines are at right angles, $(g_3g_4p'P_1)$ is harmonic by 5 *supra.*

Limiting points. This term is a little misleading. Perhaps it would be better to call them limiting circles, as they are limiting forms of circles of the system. There are three limiting circles, two of them being point circles on the line of centres, and the third a circle of infinitely large radius consisting of the radical axis and the line at infinity.

The limiting point circles when considered as points become P_2, P_3, two of the vertices of the common self-conjugate triangle. When considered as circles they coincide with their asymptotes by Art. 179, and become respectively the pairs of common chords through P_2 and P_3.

The limiting circle of infinitely large radius becomes the pair of common chords through the third vertex P_1.

The property that the radical axis bisects the segment joining the limiting point circles becomes $(pp'P_2P_3) = -1$.

Of any *pair* of common chords one can be taken to represent the radical axis, and the other to represent the line joining the circular points in the original figure.

21. The centres of similitude of two circles become a pair of tangent vertices.

22. The condition that two chords pq, $p'q'$ of a circle, centre t, are equal, is equivalent to either of the conditions

$$T(II'PQ) = T(II'P'Q'),$$

or the c.p. $$(II'PQ) = (II'P'Q').$$

23. Two orthogonal circles, centres c, t, intersecting in a, b, possess the following fundamental properties, from any one of which the others can be deduced:

(1) The tangents at a point of intersection are at right angles.

(2) The tangents at a point of intersection pass through the centres.

(3) The centre of one circle is the pole of the common chord for the other circle and the pole of ii' for its own circle.

In (1) and (2) it follows that if the property is true for one point of intersection, and in (3) for one centre, it is true for the other also, and any one of the three properties might be taken as defining two orthogonal circles.

Consequently, if we generalise two orthogonal circles we shall obtain two conics α, β which will possess the following properties:

Let II', AB be a pair of common chords, T, C the poles of II' for α, β, see Fig. 100. Then

(1) The tangents at A divide II' harmonically.

(2) The tangents at A pass through T, C.

(3) T is the pole of AB for β.

In (1) and (2) it follows that if the property is true for the point A corresponding properties hold for the tangents at B, and in (3) C is the pole of AB for α. From the property in (3) for convenience of reference we have called two conics which are so related contra-polar conics, or from the property in (1) they might be called harmotomic conics as explained in Arts. 203, 204. Hence a pair of orthogonal circles become a pair of contra-polar conics.

EXAMPLES.

In the column on the left are given the elements of the original figure which the student should draw for himself. In the column on the right will be found the corresponding elements and properties of the generalised theorems.

In every case the letters I, I' are the representatives of the circular points at infinity, and when I, I' are on the curve, T the pole of II' represents the centre of the circle.

The letters have been so chosen that Fig. 126 will apply to Examples 1—14.

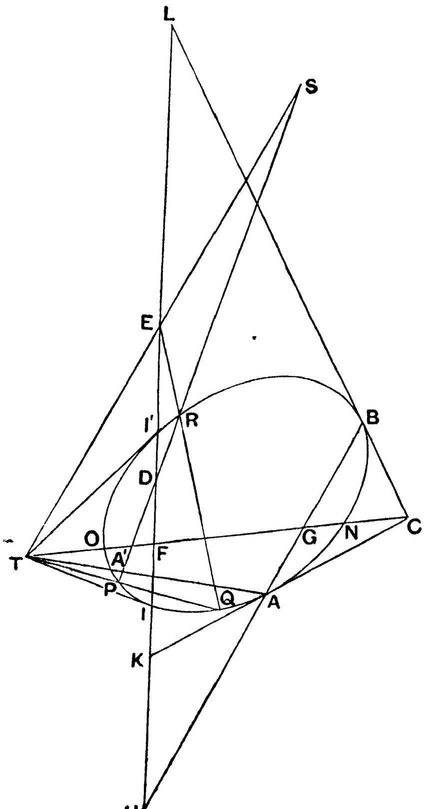

Fig. 126.

1. The angle contained in the same segment of a circle is constant.

The cross-ratio of four fixed points on a conic is constant. Art. 129.

Let a, b be two fixed points on a circle, p a variable point on it, and let pa, pb meet the line at infinity in α, β. Then by 6 *supra* $p\,(\alpha\beta ii')$ is constant. If the circle becomes a conic we have $P\,(ABII') = \text{constant}$.

2. The tangent at any point of a circle is at right angles to the radius through the point of contact.

Any chord of a conic is cut harmonically by any tangent and the line joining its point of contact to the pole of the chord. Art. 178.

The line at infinity becomes a chord cutting the conic in I, I'. The centre of the circle becomes T, the pole of II'. The radius of the circle becomes the line joining T to any point A on the curve. Then by 5 *supra* TA and the tangent at A cut II' harmonically. From this we can at once deduce that if a variable tangent meets two fixed tangents, it is divided harmonically by them, their chord of contact and the curve. Also if the tangent at A meets II' in K, TA and TK are conjugate lines since they divide II' harmonically.

3. Any diameter of a circle is bisected at the centre.

Any chord through a given point is divided harmonically by the curve, the point, and its polar. Art. 161.

4. If pq is a diameter of a circle centre t, the c.p. $(pqii')$ is harmonic.

If II' is a chord of a conic, T its pole, and TPQ a chord through T, the c.p. $(PQII')$ is harmonic, *i.e.* PQ, II' are conjugate lines. Art. 171.

Two chords are conjugate if either passes through the pole of the other. Art. 165.

5. The angle in a semi-circle is a right angle.

In Ex. 4, R is any point on the curve. If PR, QR meet II' in D, E, then $(II'DE)$ is harmonic. Art. 208 (γ).

6. If a straight line through the centre of a circle bisects a chord which does not pass through the centre, it cuts it at right angles; and conversely, if it cuts it at right angles, it bisects it.

Given a chord of a conic II', T its pole, and any chord PR cutting II' in D, and S on PR so that $(PRDS)$ is harmonic. Let TS meet II' in E. Then $(II'DE)$ is harmonic; and conversely, if $(II'DE)$ is harmonic, so also is $(PRDS)$.

Also, since $T\,(II'DE)$ and $T\,(PRDE)$ are harmonic, $T\,(PR,\ II',\ DE)$ is an involution pencil, of which TD, TE are the double rays. Art. 208 (δ).

7. *ca*, *cb* are tangents to a circle, centre *t*; *ab* and *ct* meeting in *g*.

(1) *ct* bisects the angle *atb*.

(2) *ct* bisects the angle *acb*.

(3) *ct* bisects *ab* at right angles at *g*.

II′, *AB* are two chords of a conic, *T*, *C* their poles.

(1) *TC* is one of the double rays of the involution pencil *T* (*II′*, *AB*). Art. 208 (ζ).

(2) If *CT*, *AB*, *AC*, *BC* meet *II′* in *F*, *H*, *K*, *L*, *CT* is a double ray of the involution pencil

C (*II′*, *KL*).

(3) The ranges (*ABGH*) and (*II′FH*) are harmonic.

Hence *TC* and *TH* are the double rays of the involution pencil

T (*II′*, *AB*, *KL*).

From the above we also obtain the properties:

Tangents to a conic subtend equal angles at the focus, and if *T* is the focus and *C* the pole of a chord *AB*, then *AB* is divided harmonically by *CT* and the directrix, the polar of *T*.

8. If the tangent at any point *a* of a conic meets the *S* directrix in *k*, *aSk* is a right angle.

Given a chord *II′* and its pole *T*, if the tangent at any point *A* meets *II′* in *K*, *TA* and *TK* are conjugate lines. See Ex. 2 *ad fin.*

9. *ab* is a fixed chord of a circle, *c* its pole, and *r* any point on the circumference. The bisectors of the angle *arb* pass through two fixed points, viz. the extremities of the diameter passing through *c*.

II′, *AB* are two given chords of a conic, *T*, *C* their poles. *CT* meets the conic in *N*, *O*. If *R* is any variable point on the curve, the double rays of the involution pencil *R* (*AB*, *II′*) always pass through the points *N*, *O*. Art. 164 *ad fin.*

10. Two parallel tangents to a circle intercept on any variable tangent a segment which subtends a right angle at the centre.

II′ is a given chord of a conic, *T* its pole. *K* is a point on *II′*. *KA*, *KA′* are two tangents cut at *C*, *C′* by the variable tangent at *B*. Then *T* (*II′CC′*) is harmonic. Art. 175 (1).

11. If *ca*, *cb* are tangents to a circle, centre *t*, the circle round *abc* has *ct* for a diameter.

AB, *II′* are two chords of a conic, *C*, *T* their poles. The six points mentioned lie on a conic and *CT* is the polar of the intersection of *AB* and *II′* for both conics. Art. 199.

12. Chords of a circle which subtend equal angles

(1) at the centre,

(2) at the circumference,

envelop a concentric circle.

II' is a given chord of a conic, T its pole, R, Q two points on the curve such that

(1) $T(II'RQ)$ is constant,

(2) the c.p. $(II'RQ)$ is constant,

RQ envelops a conic having double contact with the given conic at I, I'. Arts. 159, 192, 274.

13. The envelope of the chord of a conic which subtends a constant angle at the focus is a conic having the same focus and directrix; and so is the locus of its pole.

The focus and directrix become a pole T and its polar UV. Let I be on TU and I' on TV. Then corresponding to the moving chord pq of the given conic in the original figure we have a chord PQ in the generalised figure such that $T(UVPQ)$ is constant, and as in Ex. 12 (1) PQ envelops a conic having double contact with the other, and similarly for the locus of the pole of PQ.

14. In any conic the intercept on a variable tangent made by two fixed tangents subtends a constant angle at the focus.

Here I, I' are points (other than the points of contact) on the tangents from T. In Fig. 126 let a variable tangent meet the tangents from T in t_1, t_2, and those from C in c_1, c_2. Then by Data No. 6 the generalised property becomes: $T(II'c_1c_2)$ is const. *i.e.* $(t_1t_2c_1c_2)$ is const. and we have the anharmonic property of tangents, Art. 130.

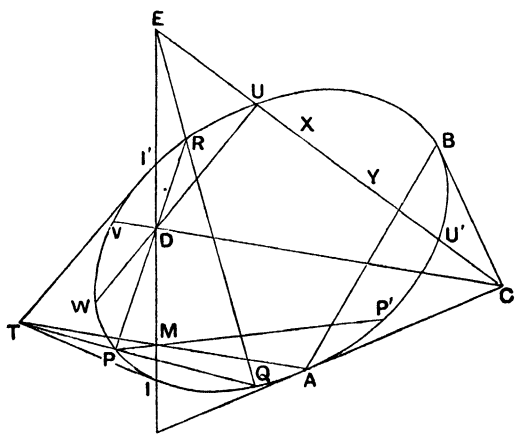

Fig. 127.

[The letters in Examples 15—19 refer to Fig. 127.]

15. If ap is any chord of a circle, pq the diameter through p, and pm the perpendicular on the tangent at a meeting the circle in p', then ap bisects the angle $p'pq$, and $p'q$ is parallel to the tangent at a.

II' is a given chord of a conic, T its pole, TPQ a chord through T. The tangent at any point A meets II' in K. M is the harmonic conjugate of K on II', PM meets the curve in P'. Then PA is a double line of the involution pencil $P(II'QP')$ and QP' passes through K.

16. The envelope of a chord uv of a circle which subtends a right angle at a fixed point c not on the curve is a conic having the fixed point and the centre of the circle for foci.

II' is a fixed chord of a conic, T its pole, and C a point not on the curve. If UV is a variable chord such that $C(II'UV)$ is harmonic, UV envelops a conic inscribed in the quadrilateral $CITI'$, and CT is a side of the common self-conjugate triangle of the two conics.

17. c is a fixed point in the plane of a circle, u any point on the curve. If ud is drawn making a right (or any constant) angle with uc, ud envelops a conic having c for focus.

II' is a given chord of a conic, C a fixed point in its plane. Any straight line through C cuts the conic in U, and II' in E, and $(II'ED)$ is harmonic (or constant). Then UDW envelops a conic touching CI and CI'.

18. c is a fixed point in the plane of a circle, centre t. The locus of the mid-points of all chords through c is the circle on ct as diameter.

II' is a chord of a conic, T its pole, C a fixed point in its plane. Any chord UU' through C meets II' in E and $(UU'EX)$ is harmonic. The locus of X is a conic through C, T, I, I' and having CT, II' conjugate chords. Art. 191.

19. The locus of the points where parallel chords of a circle are cut in a given ratio is an ellipse having double contact with the circle at the extremities of the diameter perpendicular to the chords.

If through a fixed point C a straight line is drawn meeting a conic in U, U', and on it a point Y is taken such that $(UU'CY)$ is constant, the locus of Y is a conic having double contact with the given conic at the extremities of the polar of C. Art. 273.

20. Two pairs of the lines joining the extremities of two diameters of a circle are parallel, and the other two pairs are at right angles.

II' is a chord of a conic, T its pole, TAB, $TA'B'$ are two chords through T. Then if AA', BB' meet in C, and AB', $A'B$ meet in D, the points C, D will lie on II' and divide it harmonically.

21. The locus of the mid-points of a series of parallel chords of a circle is the diameter perpendicular to the chords.

If a series of chords of a conic all pass through a point D, the locus of the harmonic conjugates of D on the chords is a straight line on which are situated the poles of all the chords. Art. 161.

Let AB be any chord through D, C its pole, and R a point on it such that $(ABDR)$ is harmonic. Let T be the pole of any chord PQ not passing through D, and let CR cut PQ in E. Then E lies on the polars of D and T. Hence any chord through E, such as PQ, is cut harmonically at E and the point where it meets DT. Therefore TE, TD are harmonic conjugates of the tangents TP, TQ. Now draw any transversal through D, cutting TP, TQ in the points I, I', and transform so that I, I' become i, i'. Then D is on the line at infinity, and CRE becomes the locus of the mid-points of a system of parallel chords of which TD is the direction. Moreover, T is now a focus, and PQ the corresponding directrix, and TD, TE being conjugate for Ti, Ti', are at right angles to each other. Hence we have the theorem: "The locus of the mid-points of a series of parallel chords of a conic is a straight line which cuts a directrix in a point E such that the corresponding focal distance TE is perpendicular to the system of chords. Also the poles of the system of chords all lie on the locus of their mid-points."

22. If a line is drawn through a focus of a central conic making a constant angle with a tangent, the locus of their intersection is a circle.

TA, TB are two given tangents to a conic, I, I' given points on them. If a variable tangent at O meets TA, TB, II' in P, Q, R respectively, and $(PQRS)$ is constant, the locus of S is a conic through I, I'.

[In the original figure let s be the focus, and let st meet the tangent at p in t so that stp is constant, and draw sy perpendicular to pt. Then the triangle syt is of constant species, and since y describes a circle, the locus of t is a circle.]

If the line II' is at an infinite distance so that R is at infinity, and PQ is divided in a constant ratio at S, the theorem becomes "The locus of the point where the intercept on a variable tangent made by two fixed tangents is divided in a constant ratio is a hyperbola whose asymptotes are parallel to the given tangents." Chap. XI, Ex. 13.

If the original conic is a parabola, the locus, instead of being a circle, is a straight line; and as II' is now a tangent, the property becomes "If a variable tangent to a conic meets three fixed tangents in P, Q, R, and $(PQRS)$ is constant, the locus of S is a straight line." If one of the tangents, as II', is at infinity, the generalised conic is a parabola, one of the three points, as R, is at infinity, and the theorem becomes "The locus of the point where the intercept on a variable tangent to a parabola made by two fixed tangents is divided in a constant ratio is a straight line." See Chap. VI, Ex. 26.

23. The locus of the intersection of tangents to a conic at right angles is a circle.

The locus of the intersection of tangents to a conic which divide a finite segment II' harmonically is a conic through I, I'.

If P is any point on the locus, PI, PI' are, by Art. 166 (β'), conjugate lines for the conic. Therefore I, I' are conjugate points, and the proposition is clearly equivalent to that of Art. 185.

If the given conic is a parabola, the segment II' is a tangent, and the locus becomes the line II' and the line joining the points of contact of tangents from I and I'.

24. The locus of the intersection of tangents to a parabola which meet at a given angle (not a right angle) is a hyperbola having double contact with the parabola.

The locus of the intersection of tangents to a conic which divide a finite segment II' of a given tangent in a constant cross-ratio is a conic having double contact with the given conic at the points of contact of tangents from I and I'.

25. If a variable triangle is inscribed in a circle, and two of its sides are parallel to given directions, the base envelops a concentric circle.

If a triangle is inscribed in a conic, having two of its sides passing through two fixed points on a given chord II', the base envelops a conic having double contact with the given conic at the points I, I'. Art. 276.

26. If two circles are concentric, any chord of one which touches the other is bisected at the point of contact.

If two conics have double contact, any chord of one which touches the other is divided harmonically by its point of contact, the chord of contact and the curve. Art. 266.

27. If two circles touch one another, the straight line joining their centres passes through the point of contact.

If two conics touch one another, the line joining the poles of their common chord passes through their point of contact.

28. If two circles, centres t, t', touch one another at p, and a chord pqr is drawn, the radii tr, $t'q$ are parallel; and conversely.

Two conics touch one another at P, and T, T' are the poles of their common chord II'. If any chord PQR is drawn through P, TR, $T'Q$ will meet on II'; and conversely, if TR, $T'Q$ meet on II', QR passes through P.

29. The locus of the centres of circles touching a given circle at a given point is a straight line passing through the given point.

A system of conics is described touching a given conic at a given point P, and intersecting it in two fixed points I, I'. The locus of the poles of II' is a straight line passing through P.

30. The centre of any circle which touches two intersecting straight lines lies on the bisector of the angle between them.

Given two tangents CA, CB and two points I, I' on a conic, the locus of the pole of the common chord II' is a double line of the involution pencil $C\,(II', AB)$. Art. 288.

31. Given the focus and two points on a conic, the directrix passes through one of two fixed points.

Given two tangents and two points on a conic, their chord of contact will pass through one of two fixed points. Art. 288.

32. The locus of the centres of circles which touch each of two parallel lines is a straight line parallel to the others and midway between them.

I, I' are two fixed points, PQ, PR two fixed lines intersecting in P on II'. If a system of conics is described passing through I, I' and touching PQ, PR, the locus of the pole of II' is a straight line through P.

33. If two circles intersect at a, b, and through a any double chord paq is drawn, pq subtends a constant angle at b.

Two conics intersect in the points I, I', A, B. Any chord through A meets the conics in P, Q. Then $B\,(II'PQ)$ is constant.

System of Co-axial circles.

34. The centres of the circles are collinear.

Pencil of conics.

The poles of any *pair* of common chords are collinear. Art. 216.

The poles lie along one of the sides of the common self-conjugate triangle $P_1P_2P_3$ (Fig. 102), which may be called the line of poles corresponding to that *pair*.

35. The line of centres is perpendicular to the radical axis.

Any *pair* of common chords forms a harmonic pencil with the two lines of poles through their vertex. Art. 214.

36. Two given points are conjugate for only one circle of a co-axial system.

Two given points are conjugate for only one conic of a pencil. Art. 217.

37. If two points are conjugate for two circles, they are conjugate for the co-axial system to which they belong.

If two points are conjugate for two conics A, B, they are conjugate for the pencil to which A and B belong. Art. 218.

38. The line of centres cuts the system in a range in involution of which the limiting point circles are the double points.

A line of poles cuts the pencil of conics in a range in involution of which a pair of vertices of the self-conjugate triangle $P_1P_2P_3$ are the double points. Art. 214.

39. A system of circles which pass through a fixed point a and have their centres collinear are co-axial, the second point b being such that ab is bisected at right angles by the line of centres.

A system of conics which pass through three fixed points A, I, I', and have the poles of II' collinear, pass through a fourth point B. If L is the line of poles, cutting II' in K, then AB is the harmonic conjugate of AK for AI, AI', and B is the harmonic conjugate of A for the points where AB cuts L and II'.

40. A system of circles which pass through a fixed point and have a pair of points conjugate are co-axial.

A system of conics which pass through three fixed points and have a pair of points conjugate form a pencil. Art. 218.

If the pair of conjugate points are i, i', the conics are all rectangular hyperbolas, and the fourth point through which they pass is the orthocentre of the triangle formed by the three given points.

41. A system of circles which have two pairs of conjugate points is co-axial.

A system of conics which pass through two fixed points and have two pairs of conjugate points form a pencil. Art. 218.

42. Three pairs of conjugate points determine a circle.

Two points and three pairs of conjugates determine a conic. Art. 218.

43. The polars of any point on the radical axis intersect on the radical axis.

The polars of a point on any common chord intersect on that common chord. Art. 210.

44. The polars of any point P will all pass through the same point P'.

The polars of any point P will all pass through the same point P'. Art. 218.

45. The radical axis bisects

Any *pair* of common chords divide harmonically

(1) The segment PP' in Ex. 44.

(1) The segment PP' in Ex. 44. Art. 218.

(2) The common tangents of any pair of circles of the system.

(2) The common tangent of any pair of conics of the pencil. Art. 213, Cor.

(3) The segment joining the limiting point circles L, L'.

(3) One of the sides of the common self-conjugate triangle $P_1P_2P_3$. Art. 214.

(4) The tangents from either limiting point circle to any circle of the system.

(4) The tangents from any vertex of the triangle $P_1P_2P_3$ to any conic of the pencil. Art. 214.

46. If three circles are co-axial a common tangent to two of them is cut harmonically by the third.

A common tangent to two conics of a pencil is cut harmonically by any other conic of the pencil. Art. 213, Cor.

47. The three radical axes of any three circles taken in pairs are concurrent in the radical centre.

If three conics have a common chord II', and the other common chord of the *pair* be drawn for each pair of conics, these latter common chords meet in a point which may be called the radical pole. Art. 215.

48. If in Ex. 47 c is the radical centre, and from c tangents are drawn to the three circles, the six points of contact will lie on a circle called the radical circle, whose centre is c.

If in Ex. 47 C is the radical pole, and from C tangents are drawn to the three conics, the six points of contact lie on a conic, which may be called the radical conic, which passes through the points I, I', and has C for the pole of II'.

Limiting point circles, L, L'.

49. (1) L, L' are inverse points for each circle.

(2) The points of contact of a common tangent of two circles of the system subtend a right angle at L and L'.

(3) If PQ is a common tangent to two of the circles, the circle on PQ as diameter passes through the points L, L'.

(4) If a transversal touches one circle at Q and cuts another at R, S, LQ bisects the angle RLS.

(5) If the transversal in (4) passes through L, $(LQRS)$ is harmonic.

Vertices of self-conjugate triangle $P_1P_2P_3$.

(1) See Ex. 38.

(2) See Ex. 45 (3).

(3) If QR is a common tangent of two conics, I, I' a pair of common points, the conic described through Q, R, I, I', having QR and II' for conjugate lines passes through two vertices of the triangle $P_1P_2P_3$.

(4) A tangent to one conic at Q cuts another in R, S. If P is a vertex of the triangle $P_1P_2P_3$, PQ is a double ray of the involution pencil $P\,(II', RS)$.

(5) If the tangent in (4) passes through a vertex P of the triangle $P_1P_2P_3$, $(PQRS)$ is harmonic.

Centres of similitude of two circles.

50. (1) The centres of similitude and the centres of the two circles form a harmonic range.

(2) The two circles and their circle of similitude are co-axial.

(3) Any transversal through a centre of similitude meets the two circles in four points which lie by pairs at the extremities of parallel radii.

Tangent vertices of two conics.

(1) In Fig. 102, g_1g_4 is a common chord, α, β its poles, T_1, T_2 the corresponding tangent vertices, $(\alpha\beta T_1T_2)$ is harmonic. Art. 229.

(2) The conic through T_1, T_2, g_1, g_4 having T_1T_2 and g_1g_4 for conjugate lines passes through the points g_2, g_3.

(3) If in Fig. 106 a transversal through T_1 meets the conic A in p, p' and the conic B in q, q', and α', β' are the poles of the common chord FF', $\alpha'p$ and $\beta'q$ meet on FF', &c. Art. 248.

Orthogonal circles.

Contra-polar conics.

51. A circle which is orthogonal to two circles has its centre on their radical axis.

A conic A which is contra-polar to two conics B, C passes through two of their common points, and the pole of the line joining them for the conic A lies on the line joining the other pair of common points of B and C.

52. A circle which is orthogonal to two circles is orthogonal to the system which is co-axial with them.

A conic which is contra-polar to two conics is contra-polar to the pencil to which they belong.

53. A system of circles which is orthogonal to two given circles is co-axial.

A system of conics which is contra-polar to two given conics forms a pencil of conics.

54. If a system of conics has its centres collinear and cuts a given circle orthogonally, it is co-axial.

If a system of conics has two common points, and the poles of the line joining them are collinear, and also a conic through them is contra-polar to the system, the conics form a pencil.

55. If a system of circles passes through a fixed point and cuts a given circle orthogonally, the system is co-axial.

If a system of conics has three points common, and is contra-polar to a given conic through two of them, the system forms a pencil.

56. If c is any point on the radical axis of a co-axial system, and tangents are drawn from c to each circle, the points of contact will all lie on a circle which cuts the system orthogonally.

If C is any point on one of a *pair* of common chords of a pencil of conics and tangents are drawn from C to each conic, the points of contact will all lie on a conic which passes through the two points where the other common chord of the *pair* intersects the conics, and is contra-polar to the pencil.

57. Given a system of co-axial circles, there exists another co-axial system such that each circle of either system cuts every circle of the other system orthogonally.

Given a pencil of conics there exists another pencil such that each conic of either pencil is contra-polar to every conic of the other. The two pencils have two points I, I' common to both.

58. In Ex. 57 the line of centres of one system is the radical axis of the other.

If QR, II' are a *pair* of common chords of the first pencil, and $Q'R'$, II' a *pair* of the second, $Q'R'$ is the locus of the poles of II' for the first, and QR the locus of the poles of II' for the second.

59. In Ex. 48, the radical circle of the three given circles cuts them orthogonally.

Given three conics which have a common chord, their radical conic is contra-polar to them.

60. If two circles cut one another orthogonally, any diameter of one is cut harmonically by the other.

If two conics are contra-polar, any chord through the pole of one is cut harmonically by the other. Art. 205.

61. Any circle through the limiting points of a co-axial system is orthogonal to the system, and conversely.

Any conic through two of the common points of a pencil and two vertices of the common self-conjugate triangle is contra-polar to the pencil.

Polar circle.

62. Given a triangle ABC, only one circle can be described for which ABC is self-conjugate. This is called the polar circle of the triangle.

Polar conic.

Given a triangle ABC and two points I, I', only one conic can be described passing through I, I' and having ABC self-conjugate. This may be called the polar conic of the triangle.

To construct the conic, join AI meeting BC in D, and take E so that $(AIDE)$ is harmonic. Then E is a point on the conic. Similarly by taking I with B and C we obtain two other points F, G on the curve. We thus obtain five points on the conic.

63. The polar circle of a triangle divides the sides harmonically, and is orthogonal to the circles on the sides as diameters.

The polar conic of a triangle divides the sides harmonically, and is contra-polar to the three conics constructed as follows.

Given a triangle ABC and two points I, I'. Describe the conic passing through A, B, I, I' and having AB and II' for conjugate lines, and similarly for the groups B, C, I, I' and C, A, I, I'. These represent the circles on AB, BC, CA as diameters. Having two points common they have a radical pole which represents the orthocentre in the original figure, and the polar conic is contra-polar to them.

Confocal conics.

64. Confocal conics cut at right angles.

Range of conics. (I, I' are now pair of opposite tangent vertices.)

In Fig. 102 $(\alpha\beta T_1 T_2)$ is harmonic. Art. 229.

65. If from a point P on a conic tangents are drawn to a confocal, they make equal angles with the tangent to the given conic at P.

Art. 228.

66. The locus of the pole of a straight line L for a system of confocals is a straight line perpendicular to L.

Art. 233.

APPENDIX II

PASCAL'S THEOREM PROVED FOR THE CONIC AND LINE-PAIR BY THE METHODS OF EUCLID AND APOLLONIUS

In Figs. 128, 129, a, b, c and a', b', c' are any six points on the conic, or by threes on the pair of lines L, L'.

α, β, γ are the intersections of $(bc', b'c)$, $(ca', c'a)$, $(ab', a'b)$.

It is required to prove that α, β, γ are collinear.

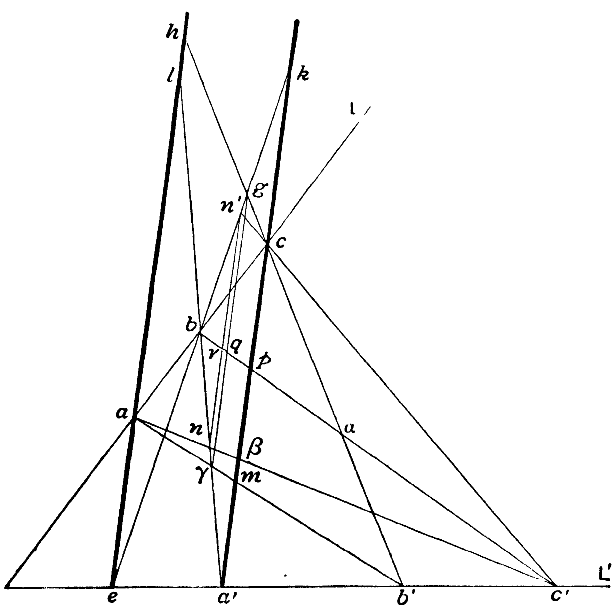

Fig. 128.

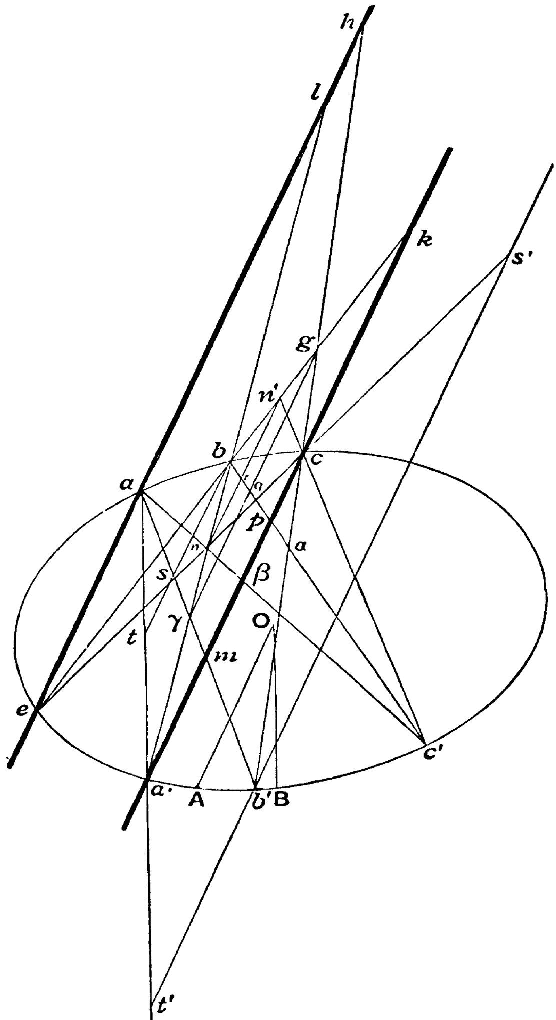

Fig. 129.

Through a draw ae parallel to ca'. In the conic, O is the centre, bst, $b's't'$, and OA are parallel to ca', and OB is parallel to aa'. Complete the figures by joining points as required.

I. For the conic.

By similar triangles

$$eh : b's' = hc : cb'$$
$$= am : mb'$$
$$= aa' : a't',$$

and
$$a'm : b't' = aa' : at',$$

$$\therefore \quad eh \,.\, a'm : aa'^2 = b's' \,.\, b't' : at' \,.\, a't'$$
$$= OA^2 : OB^2 \;\ldots\ldots\;\ldots\ldots\ldots\ldots (1).$$

Again
$$ck : sb = ec : es$$
$$= aa' : at,$$

$$\therefore \quad ck : aa' = sb : at,$$

and
$$al : aa' = bt : ta',$$

$$\therefore \quad ck \,.\, al : aa'^2 = sb \,.\, bt : at \,.\, ta'$$
$$= OA^2 : OB^2 \ldots\ldots\ldots\ldots\;\ldots (2);$$

$\therefore$ by (1) and (2)
$$ck \,.\, al = eh \,.\, a'm.$$

II. For the line-pair.

By similar triangles

$$ck : ae = kb : be$$
$$= ka' : le,$$
$$\therefore \quad ck : ka' = ae : le,$$
$$\therefore \quad ck : ca' = ae : al,$$
$$\therefore \quad ck \,.\, al = ca' \,.\, ae \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Again, from the similar triangles $b'eh$, $b'a'c$,

$$eh : a'c = b'e : b'a'$$
$$= ae : a'm,$$
$$\therefore \quad eh \,.\, a'm = a'c \,.\, ae = ck \,.\, al, \text{ by (3)},$$

$\therefore$ in both the conic and the line-pair,

$$eh : ck = al : a'm,$$

$$\therefore \quad eg : gk = l\gamma : \gamma a',$$

$$\therefore \quad ek : gk = la' : \gamma a' \quad \text{......................(4)},$$

and $$bk : be = ba' : bl,$$

$$\therefore \quad bk : ek = ba' : a'l \quad \text{......................(5)},$$

$\therefore$ by (4) and (5), $$bk : gk = ba' : \gamma a',$$

$\therefore$ γg is parallel to $a'c$ or ae.

Now γ is the intersection of $(ab', a'b)$, and g of $(eb, b'c)$.

Therefore if we introduce the point c' in the place of b', and if n be the intersection of $(ac', a'b)$, and n' that of (eb, cc'), nn' will be parallel to $a'c$ or ae, and therefore also to γg.

$$\therefore \quad \beta p : pc = nr : rn'$$

$$= \gamma q : qg,$$

$$\therefore \quad \beta p : \gamma q = pc : qg$$

$$= ap : aq,$$

$$\therefore \quad \beta p : pa = \gamma q : qa,$$

$\therefore$ a, β, γ are collinear.

INDEX

The numbers refer to pages

CAMBRIDGE: PRINTED BY JOHN CLAY, M.A. AT THE UNIVERSITY PRESS

A Selection of the Books on Mathematics, Physics and Chemistry

published by the

Cambridge University Press

A complete catalogue of the publications of the Cambridge University Press will be sent on application

ALGEBRA

Elementary Algebra. By W. W. ROUSE BALL, M.A. Extra fcap. 8vo. 4s 6d

Algebra for Secondary Schools. By CHARLES DAVISON, Sc.D. Crown 8vo. 6s. Or in two volumes, with Answers, 3s each; without Answers, 2s 6d each

Exercises separately. With Answers, 3s. Without Answers, 2s 6d

Problem Papers supplementary to *Algebra for Secondary Schools.* By C. DAVISON, Sc.D. 8d

Higher Algebra for Colleges and Secondary Schools. By C. DAVISON, Sc.D. Demy 8vo. 6s

An Algebra for Preparatory Schools. By TREVOR DENNIS, M.A. Crown 8vo. 2s. Teachers' Edition, with Answers interleaved. 5s net

Elementary Algebra. By C. H. FRENCH, M.A., and G. OSBORN, M.A. Crown 8vo. With answers, 4s 6d. Without answers, 4s

Elementary Algebra. By C. GODFREY, M.V.O., M.A., and A. W. SIDDONS, M.A.

PRICES :—	*With Answers*	*Without Answers*
Complete in one volume	4s 6d	4s
Or in two volumes :—		
Volume I, Algebra for Beginners ...	2s 6d	2s
Volume II	2s 6d	2s
Volume I is also published in two parts :—		
Part I, Chapters I—IV	1s	1s
Part II, Chapters V—XIV ...	2s	1s 6d

A School Algebra Course. By F. GORSE, M.A. With or without Answers. Crown 8vo. 3s. Or in three parts. 1s each

ARITHMETIC

The Winchester Arithmetic. By C. GODFREY, M.V.O., M.A., and G. M. BELL, B.A. Crown 8vo. 3s. Or in 2 parts, 1s 6d each. Teachers' Edition, with Answers, 6s net

Arithmetic for Schools. By C. SMITH, M.A. Extra fcap. 8vo. With or without Answers. Stereotyped Edition. 3s 6d. Or in two parts, 2s each

SOLUTIONS OF THE EXAMPLES. By G. HALE, M.A. 7s 6d

Geometrical Drawing. With Notes and Examples. By W. H. BLYTHE, M.A. Fcap. 8vo. Part I. Plane and Elementary Solid. 1s 6d. Part II. Solid or Descriptive Geometry. 1s 6d

The Contents of the Fifth and Sixth Books of Euclid. Arranged and explained by M. J. M. HILL, M.A., D.Sc., F.R.S. Second edition. Crown 4to. 6s net

Euclid. Elements of Geometry. Edited by H. M. TAYLOR, M.A.

Books I, II. 1s 6d	Books I—IV. 3s
Books III, IV. 1s 6d	Books I—VI. 4s
Books V, VI. 1s 6d	Books I—VI and XI, XII. In One Vol. complete. 5s
Books XI, XII. 1s 6d	

SOLUTIONS OF THE EXERCISES. By W. W. TAYLOR, M.A.

Books I—IV. 6s. Books VI and XI. 6s. Books I—IV, VI and XI. 10s 6d

Analytical Geometry. A First Course. By C. O. TUCKEY and W. A. NAYLER. Crown 8vo. 5s net

An Elementary Treatise on Cross-Ratio Geometry. With Historical Notes. By the Rev. JOHN J. MILNE, M.A. With 129 text-figures. Crown 8vo. 6s

Elementary Projective Geometry. By A. G. PICKFORD, M.A., M.Sc. Crown 8vo. 4s

A Course of Pure Geometry. By E. H. ASKWITH, D.D. Crown 8vo. 5s

TRIGONOMETRY

A Class Book of Trigonometry. By C. DAVISON, Sc.D. Crown 8vo. 3s

An Elementary Treatise on Plane Trigonometry. By E. W. HOBSON, Sc.D., F.R.S., and C. M. JESSOP, M.A. Extra fcap. 8vo. 4s 6d

Plane Trigonometry. By S. L. LONEY, M.A. Ninth edition. Crown 8vo. 7s 6d; or in separate Parts—Part I, an Elementary Course, excluding the use of imaginary quantities, 5s; Part II, Analytical Trigonometry, 3s 6d

SOLUTIONS OF THE EXAMPLES. Parts I and II. 6s each

The Elements of Trigonometry. By S. L. LONEY, M.A. Third edition. Extra fcap. 8vo. 3s 6d

Trigonometry for Beginners. By J. W. MERCER, M.A. Crown 8vo. 4s

Numerical Trigonometry. By J. W. MERCER, M.A. Crown 8vo. 2s 6d

Examples in Numerical Trigonometry. By E. A. PRICE, B.A. Crown 8vo. 2s

Elementary Trigonometry. By F. T. SWANWICK, M.A. Crown 8vo. 4s

Four-figure Tables. By C. GODFREY and A. W. SIDDONS. 9d net

MISCELLANEOUS TREATISES

Elementary Experimental Dynamics for Schools. By C. E. ASHFORD, M.A. Crown 8vo. 4s

The Elementary Theory of Direct Current Dynamo Electric Machinery. By C. E. ASHFORD, M.A., and E. W. E. KEMPSON, B.A. Crown 8vo. 3s net

A Primer of Astronomy. By Sir ROBERT BALL, LL.D., F.R.S. With 11 full-page Plates and 24 text-figures. Fcap. 8vo. 1s 6d net

Experimental Science. By S. E. BROWN, M.A., B.Sc. Crown 8vo. In 2 parts. Part I, Physics. 3s 6d. Also in 4 sections at 1s each. Part II, Chemistry. 2s. The two parts may be purchased together for 5s

Outlines of Chemistry. With practical work. By H. J. H. FENTON, Sc.D., F.R.S. Part I. Second (enlarged) edition. Demy 8vo. 9s net

Notes on Qualitative Analysis. Concise and Explanatory. By H. J. H. FENTON, Sc.D., F.R.S. Crown 4to. New edition, revised. 6s

First Steps in the Calculus (being Chapters XXIII—XXVI of *Elementary Algebra*). By C. GODFREY and A. W. SIDDONS. Crown 8vo. 1s 6d net

A Treatise on Elementary Hydrostatics. By J. GREAVES, M.A. Crown 8vo. 5s

SOLUTIONS OF THE EXAMPLES. Crown 8vo. 5s

Tables and Constants to Four Figures for use in Technical, Physical, and Nautical Computation, and adapted to the Requirements of Junior Mathematical Students. Compiled by W. HALL. Demy 8vo. Reprinted, with additions, 1911. 3s net

A Course of Pure Mathematics. By G. H. HARDY, M.A. Demy 8vo. 12s net

The Principles of Projective Geometry applied to the Straight Line and Conic. By J. L. S. HATTON, M.A. Royal 8vo. 10s 6d net

A Treatise on Plane Trigonometry. By E. W. HOBSON, Sc.D., F.R.S. Demy 8vo. Third edition, revised and enlarged. 12s

CAMBRIDGE TRACTS ON MATHEMATICS AND MATHEMATICAL PHYSICS

General Editors—J. G. LEATHEM, M.A., and G. H. HARDY, M.A., F.R.S.

No. 1. **Surface and Volume Integrals used in** Physics. By J. G. LEATHEM, M.A. Second edition, with two additional sections. 2s 6d net

No. 2. **The Integration of Functions of a Single** Variable. By G. H. HARDY, M.A., F.R.S. [*Out of print*

No. 3. **Quadratic Forms and their Classification** by means of Invariant Factors. By T. J. I'A. BROMWICH, M.A., F.R.S. 3s 6d net

No. 4. **The Axioms of Projective Geometry.** By A. N. WHITEHEAD, Sc.D., F.R.S. [*Out of print*

No. 5. **The Axioms of Descriptive Geometry.** By A. N. WHITEHEAD, Sc.D., F.R.S. 2s 6d net

No. 6. **Algebraic Equations.** By G. B. MATHEWS, M.A., F.R.S. 2s 6d net

No. 7. **The Theory of Optical Instruments.** By E. T. WHITTAKER, M.A., F.R.S. 2s 6d net

No. 8. **The Elementary Theory of the Symmetrical Optical Instrument.** By J. G. LEATHEM. 2s 6d net

No. 9. **Invariants of Quadratic Differential Forms.** By J. E. WRIGHT, M.A. 2s 6d net

No. 10. **An Introduction to the Study of Integral** Equations. By MAXIME BÔCHER, B.A., Ph.D. Second edition. 2s 6d net

No. 11. **The Fundamental Theorems of the** Differential Calculus. By W. H. YOUNG, F.R.S. 2s 6d net

No. 12. **Orders of Infinity: The "Infinitärcalcul"** of Paul Du Bois-Reymond. By G. H. HARDY. 2s 6d net

No. 13. **The Twenty-Seven Lines upon the Cubic** Surface. By ARCHIBALD HENDERSON, M.A., Ph.D. 4s 6d net

No. 14. **The Twisted Cubic,** with some account of the metrical properties of the Cubical Hyperbola. By P. W. WOOD, M.A. 2s 6d net

No. 5. **Complex Integration and Cauchy's Theorem.** By G. N. WATSON, M.A. 3s net

No. 16. **Linear Algebras.** By L. E. DICKSON, Ph.D. 3s net

No. 17. **The Propagation of Disturbances in Dispersive Media.** By T. H. HAVELOCK, M.A., F.R.S. 3s 6d net